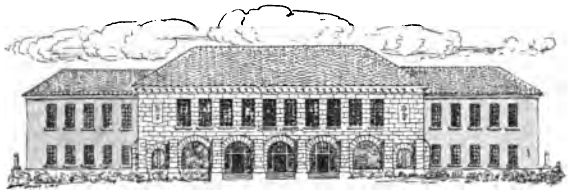

ACADEMIC ALGEBRA

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PREFACE

This work is intended to cover the subject of elementary algebra with sufficient thoroughness to prepare the student for college. It presupposes no knowledge of the subject, and it leaves for subsequent study many of the topics presented in the authors' "Elements of Algebra." At the same time it follows the latter work in the endeavor to modernize the subject and to pay relatively little attention to those portions which hold their place merely because of tradition. It is believed that teachers will welcome the logical, and at the same time simple, presentation of subjects like evolution, factoring, the theory of indices, and the treatment of the quadratic as set forth in this work. It is also felt that the time has arrived for the modern presentation of the imaginary as here given.

Many teachers wish to have a text-book with a minimum of theory simply presented, and a maximum of practical work with problems. They wish to supplement the former by explanations before the class, but they do not wish to occupy the time in dictating exercises. For such teachers this book will be found especially helpful. The problems are very numerous and are well graded, while additional work on every chapter is available in the authors' "Elements of Algebra." Teachers desiring to give lessons on

symmetry, elementary functions, graphs, equivalent equations, complex numbers, determinants, and other subjects somewhat in advance of many elementary classes, will find the material at hand in the work mentioned.

W. W. BEMAN.
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ACADEMIC ALGEBRA

CHAPTER I

INTRODUCTION TO ALGEBRA

I. ALGEBRAIC EXPRESSIONS

1. There is no dividing line between the arithmetic with which the student is familiar and the algebra which he is about to study. Each employs the symbols of the other, each deals with numbers, each employs expressions of equality, and each uses letters to represent numbers.

In arithmetic the student has learned the meaning of 2^2; in algebra he will go farther and will learn the meaning of $2^{\frac{2}{3}}$. In arithmetic he has learned the meaning of $3 - 2$; in algebra he will go farther and will learn the meaning of $2 - 3$.

In arithmetic he has said,

If $2 \times$ some number equals 10,

the number must be $\frac{1}{2}$ of 10, or 5.

In algebra he will express this more briefly, thus:

If $\qquad 2x = 10,$

then $\qquad x = 5;$

indeed he may already have met this form in arithmetic.

By arithmetic he probably could not solve a problem of this nature: The square of a certain number, added to

5 times that number, equals 50; to find the number. But after studying algebra a short time, he will find the solution quite simple.

In arithmetic it is quite common to use a letter to represent a number, as r to represent the *rate* of interest, i to represent the *interest* itself, p the *principal*, etc. In algebra this is much more common. In arithmetic it is customary to denote multiplication by the symbol $\times$, the product of 5% and \$100 being written 5% $\times$ \$100, and the product of r and p by $r \times p$; but in algebra the latter product is represented by rp.

In expressing 5 times 2 we cannot write it 52, because that means $50 + 2$. But where only letters are used, or one numeral and one or more letters, *we may define the absence of a sign to mean multiplication.* Thus, ab means $a \times b$, that is, the product of the numbers represented by a and b; $5ab$ means 5 times this product.

EXERCISE I

If $a = 5$, $b = 7$, $c = 3$, $d = 1$, $e = 4$, find the value of each of the expressions in Exs. 1–21.

1. $5abd$.
2. $\frac{2}{3}acde$.
3. $\dfrac{21ae}{4bcd}$.
4. $21bce$.
5. $225ac$.
6. $\dfrac{35ab}{3cde}$.
7. $ab - cd$.
8. $acd - e$.
9. $b - de$.
10. $abc - d$.
11. $bc - ad$.
12. $bc - de$.
13. $2a + 3b$.
14. $3b - bc$.
15. $bcd - a$.
16. $\dfrac{2a + 4d}{b}$.
17. $\dfrac{be - ad}{23c}$.
18. $\dfrac{a + d - e}{cd}$.
19. $a + b + c$.
20. $a + b - e$.
21. $a + d - c$.

If $a=2$, $b=3$, $c=4$, $d=5$, find the value of each of the expressions in Exs. 22–29.

22. $\frac{abc}{bcd}+\frac{abc}{acd}$.

23. $\frac{a+d}{7b}-\frac{c-b}{3}$.

24. $\frac{a}{b}+\frac{b}{c}-\frac{d}{bc}$.

25. $\frac{4}{a}+\frac{6}{b}-\frac{8}{c}+\frac{10}{d}$.

26. $\frac{c}{a}+\frac{3d}{b}-\frac{2b}{3a}$.

27. $\frac{a}{4}-\frac{b}{6}+\frac{c}{8}-\frac{d}{10}$.

28. $\frac{a+c}{b}+\frac{d-b}{a}-3$.

29. $\frac{a+b+c+d}{7}+\frac{5a-d}{5}$.

If $x=10$, $y=3$, $z=5$, find the value of each of the expressions in Exs. 30–39.

30. $\frac{3x}{y}+\frac{5y}{z}$.

31. $\frac{2yz}{x}+\frac{3xy}{9}$.

32. $\frac{xy-yz}{x+z}-\frac{yz-x}{y+2}$.

33. $\frac{x+z}{y}+\frac{x-z}{5}+\frac{x}{z}$.

34. $\frac{x}{z}+\frac{2y}{3}+\frac{4z}{x}+\frac{6}{y}+2$.

35. $\frac{x-y-z}{2}+\frac{x+y-z}{2}$.

36. $10xy+\frac{50z}{x}-300+xyz$.

37. $\frac{x+z}{5}+\frac{y+z}{4}+\frac{x+2y}{8}+z$.

38. $\frac{x+2y+3z}{31}+\frac{3x+2y+z}{41}$.

39. $10x+10y-10z+xy+yz$.

2. A collection of letters, or of letters and other number-symbols, connected by any of the signs of operation ($+$, $-$, $\times$, $\div$, etc.) is called an *algebraic expression.*

$2a + b$ is an algebraic expression, 2 and a being connected by the (understood) sign of multiplication, and $2a$ and b by the sign of addition. $3 + 2$ is, however, an arithmetic expression.

3. An algebraic expression containing neither the $+$ nor the $-$ sign of operation is called a **term** or a **monomial.** In the expression $2a + b$, $2a$ is a term or a monomial, and so is b.

4. An algebraic expression made up of several terms or numbers connected by the sign $+$ or $-$ is called a **polynomial.** Such an expression is $2a + b + 3c$.

5. A polynomial of two terms is called a **binomial,** one of three terms a **trinomial.** Special names are not given to polynomials of more than three terms. The expression

$\frac{2}{3}a^2 - \frac{c}{d}$ is a binomial. $5\sqrt{a} - \frac{b}{c} + ab^2cd$ is a trinomial.

EXERCISE II

1. Select the algebraic expressions in the following list:

(a) $3a^2bc$. (b) $\frac{2}{3}a^2bcd$.

(c) $2 - 3\sqrt{7} + 1$. (d) $2x^3 - 3x^2 - 9x + 1$.

2. From the following list select (*a*) the monomials, (*b*) the binomials, (*c*) the trinomials:

(a) $2a^2b$. (b) $\dfrac{ab^2}{a^2bc}$.

(c) $a^2 - 3b^2$. (d) $a^2 + 2ab + b^2$.

(e) $a - \dfrac{2ab}{3cd}$. (f) $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{2d}$.

6. In the operation of multiplication expressed by $a \times b \times c$, $a \cdot b \cdot c$, or abc, the a, b, and c are called the **factors** of the expression, and the expression is called a **multiple** of any of its factors.

Factors should be carefully distinguished from terms. The former are connected by signs of multiplication, expressed or understood; the latter by signs of addition or subtraction. In the expression $3a + bc$, the terms are $3a$ and bc; each term has two factors, those of the first one being 3 and a.

7. Any factor of an expression is called the **coefficient** of the rest of the product. The word is usually applied to some factor whose numerical value is expressed or known and which appears first in the product.

In the expression $3ax$, 3 is the coefficient of ax, and $3a$ is the coefficient of x.

Since $a = 1a$, the coefficient 1 may be understood before any letter.

8. As in arithmetic, the product of several equal factors is called a **power** of one of them. Thus, $2 \times 2 \times 2$ is called the third power of 2 and is written 2^3; $aaaaa$ is called the fifth power of a and is written a^5.

9. The number-symbol which shows how many equal factors enter into a power is called an **exponent**. Thus, in 2^3, 3 is the exponent of 2; in a^5, 5 is the exponent of a. The exponent affects only the letter or number adjacent to which it stands; thus, ab^3 means $abbb$.

The exponent should be carefully distinguished from the coefficient. In the expression $2ax^3$, 2 is the coefficient of ax^3, and $2a$ of x^3; 3 is the exponent of x.

Since x may be considered as taken once as a factor to make itself, x^1 is defined as meaning x. Hence, any letter without an exponent may be considered as having an exponent 1.

EXERCISE III

1. Name the factors in each of the following monomials:

(a) $2ab$. (b) a^2.

(c) abc. (d) $3xyz$.

2. Name the terms in each of the following polynomials, and the factors in each term:

(a) $ab + bc + ca$. (b) $a^2 + bc - abc$.

(c) $abc + bcd + c^2$. (d) $a^3 + b^3 + c^3 + d^3$.

3. Name the coefficient of x in each of the following monomials:

(a) x. (b) $2x$.

(c) $125ax$. (d) $\frac{1}{2}abcx$.

4. Name the coefficients and the exponents expressed in each of the following polynomials:

(a) $2x + 3y^2 + 4z$. (b) $3x^2 + 5y^3 + 6z^4$.

(c) $12x^2 - 13y^4 + 5z^6$. (d) $8a^2 + 9b^{10} + 18c^4$.

10. The **degree of a monomial** is the same as the number of its literal factors. Thus, a^5 is of the 5th degree, a^3b^4 of the 7th, $3abc$ of the 3d, and $5a$ of the 1st. A number, like 5, is spoken of as of *zero degree* because it has no literal factors.

The word *degree* is usually limited, however, by reference to some particular letter. Thus, while $3a^2x^3$ is of the 5th degree, it is said to be of the 3d degree in x, or of the 2d degree in a, or of zero degree in other letters.

11. Terms of the same degree in any letter are called **like terms** in that letter. Thus, $3ax^2$ and $5ax^2$ are like terms, being of the same degree in each letter. $3ax^2$ and $5bx^2$ are like terms in x.

12. The **degree of a polynomial** is the highest degree of any of its terms. Thus, $x^2 + 2x + 3$ is of the second degree in x.

13. As in arithmetic, one of the two equal factors of a second power is called the **square** (or *second*) **root** of that power, one of the three equal factors of a third power the **cube** (or *third*) **root,** one of the four equal factors of a fourth power is called the **fourth root,** etc.

In any of the problems in the early part of this work these roots can easily be found by separating the numbers into factors. Thus, the square root of 81, indicated by $\sqrt{81}$, is 9, because $9 \times 9 = 81$; the fifth root of 32, indicated by $\sqrt[5]{32}$, is 2, because $2 \times 2 \times 2 \times 2 \times 2 = 32$; and the cube root of 729 is 9, because $9 \times 9 \times 9 = 729$.

14. From what has been stated it will be seen that one of the features of algebra is the representation of numbers by letters. The advantages of this plan in the solution of problems will soon appear.

Thus, if a number is represented by n, 5 times the square of *that* number will be represented by $5n^2$. If two numbers are represented by a and b, 3 times the cube of the first, divided by 5 times the square root of the second, will be represented by $\frac{3a^3}{5\sqrt{b}}$. If a number is represented by x, the square root of the sum of the number and 3 will be represented by $\sqrt{x+3}$.

15. Those terms of a polynomial or the factors of a monomial which contain letters constitute the **literal part** of the expression. Thus, the literal part of $x^2 + 2x + 1$ is $x^2 + 2x$, and the literal part of $2x$ is x. The other term of the trinomial $x^2 + 2x + 1$ is called the numerical or arithmetic part.

EXERCISE IV

What is the numerical value of each *term* in Exs. 1–4, if $a=1,\ b=2,\ c=5,\ d=3$?

1. $ab^2c^3d^4$.

2. c^2+b^4-3a.

3. $2d^3-10c-2b$.

4. $\frac{a}{b}+\frac{c}{10}+\frac{b}{d}+\frac{a}{3}$.

5. In Exs. 1–4, what is the numerical value of each polynomial?

6. From $3ax^2$, $9mx$, $14ax^3$, ax^2, $9ax^2$, and $144x$ select the like terms.

7. From ax^2, $3bx^3$, cx^2, a^3x, and $10abx^2$ select the like terms in x or any of its powers.

8. In the following monomials name the coefficients of the various powers of x, and also the exponents of x:

(a) $\frac{x}{a}$. (b) x^5. (c) $\frac{a}{3}x^9$.

(d) $23a^2x^3$. (e) $4a^2b^3cx^9$. (f) $\frac{2}{3}a^3\sqrt{b}\,x$.

9. Express algebraically that if x^2+y^2+2xy be divided by $x+y$ the quotient is $x+y$. (Use fractional form.)

10. Express algebraically that if the sum of a^2, ab, and b^2 be divided by the square of the binomial $c-d$, the quotient is x.

11. What is the degree of the polynomial ax^2+bx+c? What is its degree in x? What is its value if $a=b=c=1$, and $x=5$?

12. In $13a^2b^3x$, what is the coefficient of x? of b^3x? of a^2b^3x? What is the degree of the expression? What is its degree in x? What is the exponent of a? of b? of x?

13. What is the meaning of the expression

$$4a^2-3b+6c-d?$$

(That from 4 times the square of a certain number there has been subtracted three times another number, etc.)

14. Also of the following expressions:

(a) $a^2+2ab+b^2$. (b) a^2-b^2.

(c) $3a^3-4\sqrt[3]{b}+\sqrt{a}$. (d) $a^3+3a^2b+3ab^2+b^3$.

15. Represent algebraically the sum of 3 times the square of a number, $\frac{2}{3}$ the cube root of a second number, and 5 times the 5th power of a third number. What is the value of the expression, if the three numbers are respectively 2, 8, 1?

Given $a=4$, $b=6$, $c=9$, $d=16$, $e=8$, find the value of each of the following expressions, and designate each as a monomial, binomial, etc.:

16. $b+\sqrt{c}$.

17. $a+\sqrt[3]{e}$.

18. $b+\sqrt{a}$.

19. $2a^2b\sqrt{c}$.

20. $\dfrac{25\,abcde}{72}$.

21. $c\sqrt{c+d}$.

22. $3+\sqrt{ac}$.

23. $e+\sqrt{ad}$.

24. $c+\sqrt[3]{2a}$.

25. a^2-b+c^2.

26. a^2+b^2-d.

27. $a^2\sqrt{a+2b}$.

28. $a+b+\sqrt{8e}$.

29. $\sqrt{d}\cdot\sqrt[3]{e}-b$.

30. $c+\sqrt{d+e+1}$.

31. $d+\sqrt{b+c+1}$.

32. $\frac{2}{3}b^3-\sqrt{c}+\sqrt[3]{e}$.

33. $a+\sqrt{c}+\sqrt[3]{4d}$.

34. $\sqrt{a}+b+d+e^2$.

35. $25\sqrt[4]{d}+a^2-\dfrac{b}{3}+5$.

II. THE EQUATION

16. An equality which exists only for particular values of certain letters representing **unknown quantities**, is called an **equation**. These particular values are called the **roots** of the equation.

Thus, $x+3=5$ is an equation because the equality is true only for a particular value of the unknown quantity x, that is, for $x=2$. This equation contains only one unknown quantity.

$2+3=5$ expresses an equality, but it is not an equation as the word is used in algebra.

17. The discovery of the roots is called the **solution of the equation**, and these roots are said to *satisfy the equation.*

Thus, if $x+5=9$, the equation is solved when it is seen that $x=4$. This value of x satisfies the equation, for $4+5=9$.

18. If two algebraic expressions have the same value whatever numbers are substituted for the letters, they are said to be **identical.**

Thus, $a^2+\frac{ab}{a}$ is identical to a^2+b, and $a+b$ to $b+a$.

An identity is indicated by the symbol $\equiv$, as in $a^2+b\equiv b+a^2$; but for simplicity the sign of equality, $=$, is used in elementary works.

19. The part of an equation to the left of the sign of equality is called the **first member**, that to the right the **second member.**

The two members are often spoken of as "the left side" and "the right side," respectively.

The extensive use of the equation is one of the characteristic features of algebra.

The importance and the treatment of the equation will best be understood by considering a few problems.

In each case we say, "Let $x =$ the number," meaning that x is to *represent* the unknown quantity.

1. *Find the number to twice which if* 3 *is added the result is* 11.

1. Let $x =$ the number.

2. Then $2x =$ twice *that* number,

and $2x + 3 =$ twice the number, plus 3.

3. The problem states that this is 11, therefore

$$2x + 3 = 11.$$

4. Subtracting 3 from these equals, the results must be equal, and

$$2x = 11 - 3, \text{ or } 8.$$

5. Dividing these equals by 2, the results must be equal, and

$$x = 4.$$

6. To see if this value of x satisfies the equation, substitute it in step 3. Since $2 \times 4 + 3 = 11$, the result is correct. This is called checking or verifying the result.

20. **A check on a solution of an equation** is such a substitution of the root as shows that it satisfies the given equation.

This substitution must always be made in the original equation. Thus, in the above solution it would not answer to substitute the root in step 4, because a mistake might have been made in getting step 4 from step 3.

21. The word *check* is also used in another sense in algebra. **A check on an operation** is another operation whose result tends to verify the result of the first. Thus, if $11 - 7 = 4$, then $4 + 7$ should equal 11; this second result, 11, verifies the first result, 4.

The secret of accurate work in algebra and in arithmetic lies largely in the continued use of proper checks.

2. *Two-thirds of a certain number, added to* 5, *equals* 17. *What is the number?*

1. Let $x =$ the number.
2. Then $5 + \frac{2}{3}x =$ two-thirds of *that* number, added to 5.
3. The problem states that this is 17, therefore
$$5 + \frac{2}{3}x = 17.$$
4. Subtracting 5 from these equals, the results must be equal, and
$$\frac{2}{3}x = 12.$$
5. Therefore, $\frac{1}{3}x = 6$

and $\frac{3}{3}x = 3 \cdot 6$,

or $x = 18.$

Check. $\frac{2}{3}$ of $18 = 12$, and $5 + 12 = 17$.

3. *If to a certain number* 35 *is added, the sum equals the sum of twice that number and* 20. *What is the number?*

1. Let $x =$ the number.
2. Then $x + 35 =$ 35 added to *that* number,
3. and $2x + 20 =$ the sum of twice the number and 20.
4. But the problem states that these are equal, therefore
$$2x + 20 = x + 35.$$
5. Subtracting x from these equals,
$$x + 20 = 35, \text{ and therefore}$$
$$x = 15. \qquad \text{(Why?)}$$

Check. (What should it be?)

From the preceding problems it will be seen that the two members of an equation are like the weights in two pans of a pair of scales which balance evenly; if a weight is taken from one pan, an equal weight must be taken from the other if the even balance is preserved; if a weight is added to one pan, an equal weight must be added to the other, and, in general, any change made in one side requires a like change in the other.

22. The axioms. There are several general statements, needed in algebra, which are so obvious that their truth may be taken for granted. Such statements are called **axioms.**

The following are the axioms most frequently met in elementary algebra.

1. *Quantities which are equal to the same quantity, or to equal quantities, are equal to each other.*

That is, if $5 - x = 3$, and $1 + x = 3$, then $5 - x = 1 + x$.

2. *If equals are added to equals, the sums are equal.*

That is, if $x = y$, then $x + 2 = y + 2$.

3. *If equals are subtracted from equals, the remainders are equal.*

That is, if $x + 2 = 9$, then $x = 9 - 2$, or 7.

4. *If equals are added to unequals, the sums are unequal in the same sense.*

"In the same sense" means that if the first was greater than the second before the addition of the equals, it is after. Thus, if x is greater than 8, $x + 2$ is also greater than 10.

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

That is, if x is less than 16, $x - 3$ is less than 13.

6. *If equals are multiplied by equal numbers, the products are equal.*

That is, if $\frac{x}{3} = 6$, $x = 3 \times 6$, or 18.

7. *If equals are divided by equals, the quotients are equal.*

That is, if $2x = 6$, $x = 6 \div 2$, or 3.

8. *Like powers of equal numbers are equal.*

That is, if $x = 5$, $x^2 = 25$. We here speak of x as a *number* because it represents one.

9. *Like roots of equal numbers are arithmetically equal.*

That is, if $x^2 = 36$, $x = 6$. The axiom says "arithmetically equal," because it will soon be found that there is an algebraic sense in which roots require special consideration.

These axioms should at once be learned by number, because they are so frequently used that references by number are necessary.

23. Stating the equation. The greatest difficulty experienced by the student in the solution of problems is in the statement of the conditions in algebraic language. After the equation is formed the solution is usually simple.

While there is no method applicable to all cases, the following questions usually lead the student to the statement:

1. *What should* x *represent?* In general, x should represent the number in question.

E.g., in the problem, "Two-thirds of a certain number, plus 10, equals 30, what is the number?" x should represent the *number*.

2. *For what number described in the problem may two expressions be found?*

Thus, in the above problem, 30 and "$\frac{2}{3}$ of a certain number, plus 10," are two expressions for the same number.

3. *What is the algebraic form of each of these?*

Thus, in the problem, "$\frac{2}{3}$ of a certain number, plus 10," is expressed by $\frac{2}{3}x + 10$, and of course 30 needs no further expression.

4. *How do you state the equality of these expressions in algebraic language?*

$$\frac{2}{3}x + 10 = 30.$$

Typical solutions. In the solution of problems involving equations, the teacher will give directions as to when the axioms should be stated in full, and as to the use of checks.

1. *Find the value of* x *in the equation* $5x - 3 = x + 7$.

1.	$5x - 3 = x + 7.$	Given
2. Therefore	$5x = x + 10.$	Ax. 2
3. Therefore	$4x = 10.$	Ax. 3
4. Therefore	$x = 2\frac{1}{2}.$	Ax. 7
Check.	$5 \times 2\frac{1}{2} - 3 = 2\frac{1}{2} + 7 = 9\frac{1}{2}.$	

2. *What is that number from two-thirds of which if* 5 *is subtracted the result is* 10*?*

1. Let	$x =$ the number.	
2. Then	$\frac{2}{3}x - 5 = 10$, by the conditions.	
3. Therefore	$\frac{2}{3}x - 5 + 5 = 15$, or $\frac{2}{3}x = 15.$	Ax. (?)
4. Therefore	$x = 22\frac{1}{2}.$	Ax. (?)

Check. $\frac{2}{3}$ of $22\frac{1}{2} = 15$, and $15 - 5 = 10$.

3. *What is that number from* 12 *times which if* 3 *is subtracted the result is* 11 *more than* 5 *times the number?*

1. Let $x =$ the number.

2. Then the problem gives 3 less than 12 times that number, or

$$12x - 3,$$

and 11 more than 5 times the number, or

$$5x + 11.$$

3. Since these are equal,

$$12x - 3 = 5x + 11.$$

4. Therefore, adding 3 and subtracting $5x$,

$$12x - 5x = 11 + 3,$$

or $7x = 14.$ Ax. 2, 3

Therefore $x = 2.$ Ax. 7

How should this result be checked?

EXERCISE V

Find the value of x in Equations 1–9.

1. $20 = 5x$.
2. $4x = 220$.
3. $x + 7 = 88$.
4. $x - 1 = 35$.
5. $5x + 2 = 30 + x$.
6. $2x + 5 = x + 20$.
7. $22x + 30 = 17x + 90$.
8. $250x - 20 = 20x + 440$.
9. $12.75x + 6.25 = 7.25x + 17.25$.

10. What number is that which divided by 3 equals 18?

11. What is the number whose half added to 16 equals 21?

12. What is the number whose twentieth part added to 10 equals 20?

13. What is that number to which if 5 is added the result is 41?

14. What is that number to which if 15 is added the result is 63?

15. What is that number from one-third of which if 27 is subtracted the result is 5?

16. There is a number by which if 9 is divided the quotient is that number. Find it.

17. The sum of two numbers is 271, and one of them is 49 more than the other; required the numbers.

18. The sum of a certain number and 9 is equal to the sum of 1 and three times that number. Find the number.

19. The sum of a certain number, twice that number, and twice this second number, is 70. What is the first number?

20. The sum of 5 and five times a number, less twice the number, equals the sum of 45 and the number; required the number.

21. A number, plus twice that number, plus three times the number, equals 9 more than twice the number; required the number.

22. The united ages of a father and son amount to 100 years, the father being 40 years older than the son. What is the age of the son?

23. A certain number, plus four times this number, plus 10, equals six times the number, less four times the number, plus 40; required the number.

24. From the sum of a certain number, twice that number, and three times the number, is subtracted 15, and the result is 4 more than five times the number; required the number.

25. A person is told to think of a number, then to double it and add 10, then to add the original number, then to subtract 4; he then says that the result is 45; what was the original number?

26. A person is told to think of a number, then to multiply it by 4 and add 12, then to add to this sum the original number, then to subtract 6; he then says that the result is 16; what was the original number?

27. Two men start in business with the same number of dollars capital; the first doubles his capital and gains $300 besides, the second adds $1000 to his capital; it is then found that the sums they have are equal; required the number of dollars with which they began.

Practical applications. The equation offers a valuable method for solving many practical problems such as the student has already met in arithmetic. A few types will now be considered.

1. *What sum of money placed at interest for* 1 *year at* $4\frac{1}{2}\%$ *amounts to* \$ 836?

1. Let $x =$ the *number* of dollars.
2. Then $x + 0.04\frac{1}{2}x =$ the number of dollars in the principal + the interest.
3. But $836 =$ the number of dollars in the principal + the interest.
4. Therefore $x + 0.04\frac{1}{2}x = 836.$
5. Or $1.04\frac{1}{2}x = 836.$
6. Therefore $x = 800.$ Ax. 7
7. Therefore the sum is \$800.

Check. $800 + 0.04\frac{1}{2}$ of $800 = 836.$

It should be noticed that since x stands for the *number* of dollars, when it is found that $x = 800$ it is known that the result is \$800.

In the applied problems of algebra, x *is always taken to represent an abstract number,* and the first step should always state definitely to what this abstract number is to refer.

2. *A commission merchant sold some produce on a commission of* 2%, *and paid* \$ 5 *for freight and cartage, remitting* \$ 117.50. *For how much did he sell the produce?*

1. Let $x =$ the *number* of dollars received.
2. Then $x - 0.02x =$ the number after deducting 2%.
3. And $x - 0.02x - 5 =$ the number after deducting for cartage also.
4. Therefore $x - 0.02x - 5 = 117.50.$
5. Therefore $0.98x = 122.50.$ (Why?)
6. Therefore $x = 125.$ (Why?)

Check. $125 - 0.02$ of $125 - 5 = 117.50.$

EXERCISE VI

1. In how many years will \$100 double itself at 5% simple interest?

2. In how many years will \$600 gain \$144 at 6% simple interest?

3. \$80 is the simple interest on \$6400 for how many months at $\frac{1}{8}$% per month?

4. What sum of money put at simple interest for 2 years at 6% amounts to \$84?

5. In how many years will a sum of money double itself at 6% simple interest?

6. At what rate per cent will \$1600 gain \$640 in 10 years at simple interest?

7. In how many years will \$500 amount to \$560 at 4 per cent simple interest?

8. At what rate per cent will \$750 amount to \$795 in 2 years at simple interest?

9. What sum of money will amount to \$324 in 2 years at 4 per cent simple interest?

10. In how many years will \$80 amount to \$200 at 6% simple interest? ($80 + x \times 6\%$ of $80 = 200$.)

11. What is the rate per cent of premium for insuring a house for \$2000, when the premium is \$30?

12. Taking the circumference of a circle as being $3\frac{1}{7}$ times the diameter, find the diameter of the circle whose circumference is $7\frac{6}{7}$ ft.

13. From Ex. 12, find the radius of a circle whose circumference is $31\frac{3}{7}$ ft.

14. After selling some goods on 5% commission, a merchant remits, as the net proceeds, \$79.80. How much is his commission? (Let $x =$ the *number* of dollars for which the goods were sold; after finding x take 5% of it.)

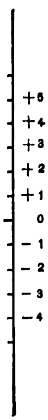
III. THE NEGATIVE NUMBER

24. In remote times people could count only by what are often called *natural numbers*, that is, 1, 2, 3, 4, 5, $\cdots$. Such numbers suffice to solve an equation like $x - 3 = 0$, an equation in which x must evidently be 3.

Mankind then introduced the *unit fraction*, that is, a fraction with the numerator 1. Such numbers are necessary in solving an equation like $2x - 1 = 0$, where $2x = 1$, and $x = \frac{1}{2}$.

Then came the *common fraction* with any numerator, as $\frac{2}{3}$, $\frac{5}{7}$, $\frac{10}{11}$, $\cdots$. Such numbers are necessary in solving an equation like $3x - 2 = 0$, where $3x = 2$, and $x = \frac{2}{3}$.

The idea of a number was then enlarged to cover the cases of $\sqrt{2}$, $\sqrt{7}$, $\sqrt[3]{5}$, $\cdots$, which are neither integers nor fractions with integral terms. Such numbers are necessary in solving an equation like $x^2 - 2 = 0$, where $x^2 = 2$, and $x = \sqrt{2}$.



25. Many centuries later the necessity was felt for further enlarging the idea of number in order to solve an equation like $x + 1 = 0$, or $x + a = 0$, a being one of the kinds of number above mentioned. This led to the consideration of *negative numbers*, $-1, -2, -3, \cdots$, and the meaning of these numbers will now be investigated.

26. If the mercury in a thermometer stands at 5° above a fixed point and then falls 1°, we say that it stands at 4° above that point. If it falls another degree, we say that it stands at 3° above that point, and the next time at 2°, and the next time at 1°.

If the mercury then falls another degree, it becomes necessary to name the point at which it stands, and we call this point **zero** and designate it by the symbol 0.

If the mercury falls another degree, we must again name the point at which it stands, and instead of calling this point "1° below zero," we call it "minus 1°" or "negative 1°," and we designate it by the symbol $-1°$. Likewise, if the mercury falls 1° lower, we say that it stands at $-2°$, and so on.

27. Thus we find a new use for the word *minus* and the symbol $-$. Heretofore both the word and the sign have indicated an *operation,* subtraction; they now indicate the *quality* of a number, showing on which side of zero it stands, and thus they are *adjectives.*

Similarly in speaking of "west longitude," "west" is an adjective modifying "longitude"; in speaking of "minus latitude," "minus" is an adjective modifying "latitude"; so in "minus 2°," "minus" is an adjective.

It is customary in America to speak of *west* longitude and of *north* latitude as *positive* (plus). Therefore, if we are speaking of longitude,

$$+10° \, 30'$$

means 10° 30′ west, the $+$ being used as an adjective.

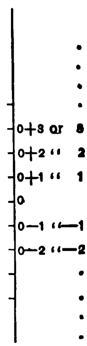
In the same way, if we are speaking of latitude,

$$-25° \, 20'$$

means 25° 20′ south, the $-$ being used as an adjective.

28. It thus appears that our idea of number can be enlarged to include zero, and still further to include the series of natural numbers extended downward from zero.

If necessary to distinguish 1° above 0 from 1° below 0, the former is written $+1°$ and called either "plus 1°" or "positive 1°," and the latter is written $-1°$. But *unless the contrary is stated, a number with no sign before it is considered positive.*



29. It thus appears that *positive numbers may be represented as standing on one side of zero, and negative numbers on the other.*

Thus, if west longitude is called positive, east longitude is called negative, and *vice versa*; if north latitude is called positive, south latitude is called negative; if a man's capital is called positive, his debts are called negative, etc.

E.g., if the longitude of New York is $73° 58' 25.5''$ west and that of Berlin is $13° 23' 43.5''$ east, the former may be designated as $+73° 58' 25.5''$ and the latter as $-13° 23' 43.5''$, their difference being $87° 22' 9''$.

Similarly, if a man begins the year with \$ 5000, and during the year loses his capital and gets \$ 2000 in debt, he is \$ 7000 worse off than at the beginning. It may then be said that he started with \$ 5000 and ends with $-$\$ 2000, the difference being the \$ 7000 which he lost.

30. Since two such expressions as $+a$ and $-a$, or $+5°$ and $-5°$, represent different directions, but equal measures, they are said to have the same **absolute value**.

Since the difference between $-5°$ and $+5°$ on a thermometer is $10°$, it appears that *we sometimes find the difference between two numbers by adding absolute values.*

31. There are numerous signs used in algebra, as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, exponents, etc. But *by the sign of a term is always meant the* $+$ *or* $-$ *sign, which indicates the quality of the term,* whether positive or negative.

Thus, in $a^2 \div 7b$, the sign of $7b$ is plus (understood), while in $-7b \div a^2$ it is minus.

32. Positive and negative numbers, together with zero, are often called **algebraic numbers**, positive numbers being called **arithmetic**.

EXERCISE VII

1. What is the difference in longitude between $-15°$ and $+75°$?

2. What is the difference in latitude between $+10°$ and $-20°$? between $+90°$ and $-90°$?

3. What is the difference in time between 6 days before January 1 and 14 days after that date?

4. What is the difference in time between 50 years B.C. and 50 years A.D.? Indicate this by symbols.

5. Draw a line representing a thermometer scale; mark off $0°$, $30°$, $-25°$. What is the difference between $30°$ and $-25°$?

6. If the upward pull of a toy balloon is represented by $+3$ lbs., what will represent the *upward* pull of a piece of iron weighing 10 lbs.?

7. Suppose the piece of iron and the balloon mentioned in Ex. 6 were fastened together. What would be their combined weight?

8. If a man's assets are represented by a line 1 in. long to the right of 0, how would his debts, which are half as much, be represented?

9. What is meant by saying that a person is worth $-\$1000$? Suppose $\$2000$ is added to his capital. How much is he then worth?

10. What is the difference in temperature between $70°$ above zero and $10°$ below zero? How should each of these temperatures be represented?

11. A ship in $8°$ west longitude $(+8°)$ sails so as to lose $1°$ in longitude. On what meridian is it then? Suppose it loses $7°$ more? $3°$ after that?

IV. THE SYMBOLS OF ALGEBRA

33. As already seen, algebra employs the symbols of arithmetic, often with a broader meaning, and introduces new ones as occasion demands. The following classification will enable the student to review the symbols thus far familiar to him, and may add a few new ones to his list. Others will be considered from time to time as needed.

1. **Symbols of quantity.**

a. *Arithmetic numbers, i.e.,* positive integers and fractions.

b. *Algebraic numbers,* the above with the addition of negative numbers and zero. Others will be considered later.

c. *Letters* denoting algebraic numbers; these are the symbols of quantity chiefly used in algebra.

2. **Symbols of quality.**

The symbols $+$ *and* $-$ indicate positive and negative number, as in $+a$, $-b$, etc.

3. **Symbols of operation.**

a. *Addition,* $+$.

b. *Subtraction,* $-$.

c. *Multiplication,* $\times$, $\cdot$, and the absence of sign. Thus, $a \times b$, $a \cdot b$, and ab, all indicate the product of a and b. It is quite customary in algebra to say "a into b" for "a times b."

d. *Division,* $\div$, $/$, $:$, and the fractional form. Thus, $a \div b$, a/b, $a:b$, and $\frac{a}{b}$, all mean the quotient of a divided by b.

In arithmetic the symbol : is used only between numbers of the same denomination; but in algebra, where the letters represent abstract numbers, this distinction does not enter.

For ease in typesetting the symbol / is often used in print; in writing, the fraction is usually employed.

e. *Involution* is indicated by exponents. *Evolution* is indicated, as in arithmetic, by the symbol $\sqrt{\ }$, a contraction of r, the initial of *radix* (Latin, root). Thus, a^3 means aaa, $\sqrt[3]{8}$ means one of the three equal factors of 8, and hence, $\sqrt[3]{8}=2$.

4. Symbols of relation.

a. *Equality*, $=$.

b. *Identity*, $\equiv$; thus, $a \equiv a$, read "a is identical to a." Also read "stands for," as in $r \equiv$ rate, $P \equiv$ principal. The sign of equality is commonly used in these cases, however.

c. *Inequality*: $>$ greater than, $<$ less than, $\neq$ not equal to, $\ngtr$ not greater than, $\nless$ not less than.

5. Symbols of aggregation.

The expression $m(a+b)$ means that $a+b$ is to be multiplied by m. The parenthesis about $a+b$ is called a *symbol of aggregation.*

The bar, brackets, and braces are also used, as in

$$m\{a-[b+c]\},$$

where $b+c$ is to be subtracted from a, and the result multiplied by m, and in $a \cdot \overline{b+c}$, which means the same as $a(b+c)$, but the term *parenthesis* is often employed to mean any symbol of aggregation. The subject is more fully discussed on p. 40.

6. Symbols of deduction.

$\because$, since.

$\therefore$, therefore.

7. Symbol of continuation.

$\cdots$, meaning "and so on," as in the sentence, "consider the quantities a, a^2, a^3 $\cdots$."

34. Order of operations. Mathematicians have established a custom as to the order in which these signs shall be considered when several are involved, as in expressions like

$$2 + 3 \times 4 \div 2 - 2 \cdot 2^2,$$

and
$$a + b \times c \div a - 2a^2.$$

It is agreed that *the operations of multiplication and division shall be performed first*, in the order in which they are indicated, and *then the operations of addition and subtraction* in their order. Hence,

$$\begin{aligned} 2 + 3 \times 4 \div 2 - 2 \cdot 2^2 &= 2 + 12 \div 2 - 2 \cdot 4 \\ &= 2 + \ 6 - 8 \\ &= 0. \end{aligned}$$

This conventional order can, of course, be varied by the use of symbols of aggregation. *E.g.*, $2 + 3 \times 5 = 17$, but $(2 + 3) \times 5 = 25$.

There are also certain exceptions to this order, but they are not of a nature to cause any confusion. *E.g.*, $ab \div cd$ means $(ab) \div (cd)$ and not $\frac{abd}{c}$. Similarly, when the sign of ratio (:) appears in a proportion it has not the same weight as the symbol $\div$. Thus, $2 + 3 : 12 - 2 = 1 : 2$ means $(2 + 3) : (12 - 2) = 1 : 2$.

EXERCISE VIII

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the value of each of the expressions 1–8.

1. $(a + b^2)^2$.
2. $b(c + d)^2$.
3. $5d/bc - a$.
4. $(9 - d \div b)c^2$.
5. $3a + b \times c - d$.
6. $(a + b)(c + d)$.
7. $2 + a^2b^3d \div a + b$.
8. $2a \times b \div d \times c - a$.

Read expressions 9–13.

9. $a + a^2 \equiv a + a^2.$

10. $a/b \not> a$ if $b > 1.$

11. $a^3 + a^2 \neq a^2$, and $a^2 + a^3 \not< a^2.$

12. $a^2 = a + a^2 - a, \therefore a^2 < a + a^2.$

13. $\because a = 2, \therefore a^2 = 4, a^3 = 8, a^4 = 16, \cdots.$

Show that the expressions in Exs. 14–19 are equal when $a = 2$ and $b = 3$. That is, substitute 2 for a and 3 for b in each member.

14. $(a + b)^2 = a^2 + 2\,ab + b^2.$

15. $(b - a)^2 = b^2 - 2\,ba + a^2.$

16. $(b^2 - a^2)/(b - a) = b + a.$

17. $(a + b)(a^2 - ab + b^2) = a^3 + b^3.$

18. $(b^3 - a^3)/(b - a) = b^2 + ba + a^2.$

19. $(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$

If $w = 3$, $x = 5$, $y = 1$, $z = 4$, find the value of each of the following expressions:

20. $10\,w + 9\,x \div 5\,y + z.$

21. $\dfrac{w + x}{z} + \dfrac{x + y}{w} + \dfrac{y + z}{x}.$

22. $\dfrac{w + x + y}{3} + \dfrac{x + y + z}{5}.$

23. $x + 3\,y + 4\,z \div 2\,y \times x - y.$

24. $8\,z - x \div y \times w + x + y - z.$

25. $w + x \div 5\,y - z + 10\,y \div x + w.$

26. $\dfrac{w + x + y + z}{y} + \dfrac{2\,w + 3\,x + 4\,y}{5}$

V. PROPOSITIONS OF ALGEBRA

35. A **proposition** is a statement of either a truth to be demonstrated or something to be done. *E.g.*, algebra investigates this proposition: The product of a^m and a^n is a^{m+n}. It also considers such statements as this: Required the product of $a+b$ and $a-b$.

36. Propositions are divided into two classes, **theorems** and **problems**.

A **theorem** is a statement of a truth to be demonstrated. *E.g.*, the product of a^m and a^n is a^{m+n}.

A **problem** is a statement of something to be done. *E.g.*, required the product of $a+b$ and $a-b$.

A **corollary** is a proposition so connected with another as not to require separate treatment. The proof is usually substantially included in that of the proposition with which it is connected.

REVIEW EXERCISE IX

1. What is the degree of the expression $3ax^2y^3$? What is its degree in x? in y? in x and y? in z?

2. Distinguish between coefficient and exponent. What is the coefficient of x in the expression $3x^4$? the exponent?

Show that if $a=3$, $b=2$, $c=1$, the statements in Exs. 3–8 are true.

3. $(a+b)(a-b)=a^2-b^2$.

4. $(a-b)^2=a^2+b^2-2ab$.

5. $(a+b)^3-(a^3+b^3)=3ab(a+b)$.

6. $(a-b)^3=a^3-b^3-3ab(a-b)$.

7. $(a+b)^2-c^2=(a+b+c)(a+b-c)$.

8. $(a^2+b^2)^2=a^4+2a^2b^2+b^4$.

9. Solve the equation $3x + 4 = 2x + 20$.

10. Solve the equation $19x - 15 = 17x + 45$.

11. Solve the equation $47x - 17 = 67 - 37x$.

12. What meaning has the number "minus 2" to you?

13. How many terms in the equation $2x^2 + 3x - 4 = 1$? How many members?

14. Draw a diagram illustrating the fact that the absolute value of the difference between -5 and 10 is 15.

15. What is the degree of the polynomial $x^3 + 3x^2y^2 + 3xy^2 + 5y + 6$? What is the degree in x? in y? in z?

16. Write the following in algebraic language: The sum of the square of a number, 3 times the number, and 5, is equal to 9.

17. Represent algebraically the sum of the cube of a number, 5 times the square of the number, and 6, less half the number.

18. What is meant by solving an equation? by a root of an equation? by checking a solution? Illustrate with the equation $x - 2 = 0$.

19. What is the number from which if 5% of it be taken, and 10% from the remainder, and 20% from that remainder, the result is 41.04?

20. If a ten-pound weight is represented by $+10$ lbs., how shall we represent the weight of a balloon that pulls upward with a force of 300 lbs.?

21. If the weight of a piece of iron is represented by $+10$ lbs., what will represent the weight of a toy balloon which pulls up with a force of 3 lbs.?

22. If 1900 years A.D. is represented by $+1900$ yrs., how should 100 years B.C. be represented? What is the difference in time between these two periods?

CHAPTER II

ADDITION AND SUBTRACTION

I. ADDITION

37. In elementary arithmetic the word **number** includes only positive integers and fractions, or at most a few indicated roots like $\sqrt{2}$, $\sqrt[3]{5}$, $\cdots$. Hence, the word **sum**, as there used, applies only to the result of adding two positive numbers.

In algebra the word **sum** has a broader meaning, and includes the results of adding negative numbers and numbers some of which are positive and others negative, these numbers being called **addends**, as in arithmetic. *E.g.*, consider the combined weight of these three articles: a 2-lb. weight, a 4-lb. weight, and a balloon which weighs -5 lbs. (*i.e.*, pulls upward with a force of 5 lbs.). Together they would evidently weigh 1 lb. Hence 1 lb. is said to be the sum of 2 lbs., 4 lbs., and -5 lbs.

So the result of adding a debt of \$100 to a capital of \$300 is a capital of \$200; hence, \$200 is said to be the sum of \$300 and $-$ \$100.

38. In this broader view of addition two cases evidently arise:

1. Numbers with like signs.

$$2 \text{ lbs.} + 3 \text{ lbs.} = 5 \text{ lbs.}$$

A balloon pulling up 5 lbs. and one pulling up 8 lbs. together pull up 13 lbs., or $(-5 \text{ lbs.}) + (-8 \text{ lbs.}) = -13 \text{ lbs.}$

2. Numbers with unlike signs.

A balloon pulling up 5 lbs. and a weight of 2 lbs. together pull up 3 lbs., or -5 lbs. $+2$ lbs. $=-3$ lbs.

39. From considerations like these we are led to define **the sum of two algebraic numbers** as follows:

1. *If two numbers have the same sign, their algebraic sum is the sum of their absolute values, preceded by their common sign.*

Thus, to add -3 and -2 means to add 3 and 2 and to place the sign $-$ before the result.

2. *If they have not the same sign, their algebraic sum is the difference of their absolute values, preceded by the sign of the one which has the greater absolute value.*

Thus, to add -3 and 2 means to find the difference between 3 and 2 and to place the sign $-$ before the result. To add 3 and -2 means to find the difference between 3 and 2 and to place the sign $+$ before the result.

40. In the special case where the two numbers are equal and of opposite signs, the sum is *zero*. *E.g.*, $2+(-2)=0$.

If one of two numbers is zero, their algebraic sum is the other number. Thus, $-3+0$ means -3.

The algebraic sum of several numbers is defined as the sum of the first two plus the third, that sum plus the fourth, $\cdots$. Thus, $2+3+4$ means $(2+3)+4$.

41. Just as 2 dollars + 3 dollars = 5 dollars, so $2d+3d=5d$. The d in this case is called the **unit of addition**. So in the case of $10x^2y$ and $-5x^2y$, the unit of addition is x^2y, and the sum is $5x^2y$.

Hence, *to add monomials having the same unit of addition we may add their coefficients*, prefixing the sum to the unit of addition.

EXERCISE X

1. Find the sum of -20, $+3$.

2. Also of $+2$, -3, $+5$, -4, $+9$.

3. Also of $2x^2$, $5x^2$, $-6x^2$, $8x^2$, $-9x^2$.

4. Also of $127\,mn$, $62\,mn$, $-93\,mn$, $-17\,mn$.

5. $\$50 + \$7 + (-\$21) + (-\$30) = ?$

6. $5 + 219 + (-376) + (-40) + 10 + (-37) = ?$

7. $3a + (-2a) + (-5a) + 8a + 6a + (-10a) = ?$

8. $(-7) + 4 + (-2) + 18 + 13 + (-20) + (-6) = ?$

9. What is the sum of $3a$, $5a$, $-6a$, $8a$, $10a$, $-3a$, $-17a$?

10. $12x^2y + 4x^2y + (-16x^2y) + (-3x^2y) + 10x^2y + x^2y = ?$

11. 5 lbs. + 55 lbs. + (– 40 lbs.) + (– 27 lbs.) + 121 lbs. + (– 19 lbs.) + (– 5 lbs.) = (?) lbs.

12. What is the combined weight of two balloons weighing, respectively, – 10 lbs. and – 18 lbs., and three pieces of iron weighing, respectively, 6 lbs., 12 lbs., and 14 lbs.?

13. On seven consecutive midnights in January, in Montreal, the temperature was 30°, 18°, 10°, 4°, 0°, – 7°, – 20°. What was the average midnight temperature for the week?

14. What is the combined weight, under water, of a piece of cork weighing – 2 oz., a stone weighing 3 lbs., a piece of wood weighing – 1 lb. 3 oz., and a piece of iron weighing 5 lbs.?

15. A merchant finds that he has cash in bank \$575.50, stock worth \$4875, due from customers \$1121.50, that he owes a note and interest amounting to \$350.25 and bills amounting to \$827, and that he owns a bond and mortgage of \$1000. Express his capital as the sum of these various items with their proper signs.

42. We have seen (§ 41) how to add algebraic monomials. It is evident that we may add polynomials in much the same way, by adding the coefficients of the same units of addition.

$$\begin{array}{r}
2a + b - 3c \\
4b + c \\
-6a - b + c \\
\hline
-4a + 4b - c
\end{array}$$

In the above case, taking a for the first unit of addition, we have $2a+(-6a)=-4a$ (§ 39, 2, and § 40); then $b+4b+(-b)=4b$, and $-3c+c+c=-c$.

If the letters are not arranged in the same order in the addends, it is evidently better to make such an arrangement so as to have like units of addition in the same column.

Hence, it appears that *to add like terms is to add the coefficients, and to add polynomials is to add their like terms,* the literal parts being properly inserted in the sum.

The sum is supposed to be simplified as much as possible. Thus, the sum of $4a-b$ and $b+a$ is $5a$, not $4a+a$.

EXERCISE XI

Add the polynomials in the following exercises:

1. $m+3n+s,\ m-s.$
2. $p+2q-r,\ 2p-3q.$
3. $2a-b-c,\ a+b+c.$
4. $3x-4y+z,\ 4x+5y.$
5. $a+3b-2c,\ -b+4c.$
6. $-x+2y-z,\ x-y+z.$
7. $x+y+15z,\ x-y-14z.$
8. $a+b,\ b+c,\ c+d,\ d+a.$
9. $2x+3a+m,\ 2y-3a-m.$

10. $2m + n$, $2n + m$, $3m - 3n$.

11. $9p + 4q - 5r$, $p + 6q + 15r$.

12. $x^2 + 2xy$, $2x^2 - 3xy$, $x^2 + xy$.

13. $a - 3b + 17c$, $5a + 9b - 11c$.

14. $ab^2c + abc^2 + a^2bc$, $-a^2bc - ab^2c$.

15. $4e + 5f + 6g$, $-4e + 5f - 6g$.

16. $2a^2 - 3b^2 + 4c$, $2a^2 - 4c + 5b^2$.

17. $2ax + 3by + 4cz$, $-6by - 3cz$.

18. $ax + mn + pq$, $3ax - 2mn - pq$.

19. $3a^2 + 4b^2 - 5c^2$, $3a^2 - 3b^2 + 6c^2$.

20. $2m - 3n + q$, $2p - 2m + 3n + q$.

21. $8x - 7y + 6z$, $9w + x + 16y + 3z$.

22. $mn^2 + n^3$, $m^3 + m^2n$, $-2m^2n - 2mn^2$.

23. $23x^2y - 17xy^2 + z$, $2z - 20x^2y + 20xy^2$.

24. $2mn + 3n^2$, $4mn - 5n^2$, $-6mn + 2n^2$.

25. $10a^3b - 14ab^3 + a^4$, $5a^4 + 15ab^3 - 10a^3b$.

26. $a + b + c$, $a - b + c$, $a + b - c$, $b + c - a$.

27. $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$, $-2x^2 + 2y^2$.

28. $12m - 13n + 11p$, $-11m + 12n - 13p$.

29. $34ax - 75by + 60cz$, $16ax + 25by - 10cz$.

30. $17a^2 - 18am + m^2$, $-15a^2 + 16am + 3m^2$.

31. $a^3 + b^3$, $b^3 + a^2b + ab^2$, $-3a^2b + b^3 - ab^2$, a^3.

32. $u^2 + 2uv$, $v^2 - 2uv$, $u^2 + 2uv + v^2$, $u^2 - 2v^2$.

33. $225a + 175b - 300c$, $-205a - 155b + 180c$.

34. $142x^4 + 31x^3y + 9x^2y^2 + 10xy^3 - 15y^4$, $-130x^4 - 30x^3y + 2x^2y^2 - 8xy^3 + 20y^4$, $-11x^4 - 10x^2y^2$.

43. Check of arbitrary values. The sum in addition is easily checked by substituting any convenient values for the letters. Thus in adding $2a^2 + b$ and $a^2 - 3b$, we may let $a = 2$ and $b = 1$, for example. We then see that

$$\begin{array}{lllr} & 2a^2 + b & \text{becomes } 2 \cdot 2^2 + 1 = & 9 \\ & \underline{a^2 - 3b} & \text{"} \qquad\quad 2^2 - 3 = & \underline{1} \\ \text{Then} & 3a^2 - 2b & \text{must become} & 10 \end{array}$$

which it does, for $3 \cdot 2^2 - 2 = 12 - 2 = 10$.

Since the sum must be true for any values of the letters, we may arbitrarily select any values we please. This simple check is therefore called the *check of arbitrary values.* Students would do well to use this in all examples in addition, subtraction, multiplication, and division, since it is quite sure to show any errors.

EXERCISE XII

Add the polynomials in the following exercises, checking the work as directed:

1. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $a^2 + b^2$. Check by letting $a = 1$, $b = 1$.

2. $3x^2 + 2xy + 4y^2$, $4x^2 - 3xy - 2y^2$, $3x^2 + xy$. Check by letting $x = 1$, $y = 1$.

3. $6m + x$, $5m - x - 3y$, $8m - 2y$, and $3x$. Check by letting $m = 4$, $x = 1$, $y = 1$.

4. $2a + 3b - c$, $-4c$, $7a$, $-6b + 8c$, and $-a + b - c$. Check by letting $a = 1$, $b = 1$, $c = 1$.

5. $17x - 9y$, $3z + 14x$, $y - 3x$, $x - 17z$, and $x - 3y + 4z$. Check by letting $x = 1$, $y = 2$, $z = 3$.

6. $16m + 3n - p$, $p + 4q$, $-q + 7m - 3n$, $n - q$, and $3n + 2p$. Check by letting $m = 1$, $n = 1$, $p = 2$, $q = 4$.

II. SUBTRACTION

44. Subtraction is the operation by which, having the sum of two expressions and one of them given, the other is found.

The given sum is called the **minuend**, the given addend is called the **subtrahend**, and the addend to be found is called the **difference** or the **remainder.**

That is, the difference is the number which added to the subtrahend produces the minuend. In other words,

$$\textit{difference} + \textit{subtrahend} = \textit{minuend}.$$

E.g.,
$$\begin{array}{llll} \because 4+5 & =9, & \therefore & 4=9-5; \\ \because 4+(-3) & =1, & \therefore & 4=1-(-3); \\ \because 4+(-5) & =-1, & \therefore & 4=-1-(-5); \\ \because -4+(-3) & =-7, & \therefore & -4=-7-(-3). \end{array}$$

These results are illustrated as follows: the difference between the temperature of 9° and that of 5° is 4°; that between 1° and $-3°$ (*i.e.*, 1° above 0 and 3° below 0) is 4°; that between $-1°$ and $-5°$ (*i.e.*, 1° below 0 and 5° below 0) is 4°; that between $-7°$ and $-3°$ is $-4°$, that is, the mercury must *fall* 4° from $-3°$ to reach $-7°$.

We may, therefore, think of subtraction as the inverse of addition, or the process which undoes addition.

Consider, for example, the following:

$$\begin{array}{ll} \text{From} & 5a+3b \\ \text{to subtract} & \underline{2a+b} \\ & 3a+2b \end{array}$$

What must be added to $2a$ to make $5a$? Evidently $3a$. What must be added to b to make $3b$? Evidently $2b$. Therefore the remainder is $3a+2b$.

Again, consider the following:

$$\begin{array}{ll} \text{From} & 4a^2 - 5ab + 2b^2 \\ \text{to subtract} & \underline{3a^2 + 4ab - 5b^2} \\ & a^2 - 9ab + 7b^2 \end{array}$$

What term added to $3a^2$ makes $4a^2$? Evidently a^2.

What term added to $4ab$ makes $-5ab$? Evidently $-9ab$; for the addition of $-4ab$ makes 0, and the further addition of $-5ab$ makes $-5ab$.

Similarly, $7b^2$ is the term which added to $-5b^2$ makes $2b^2$.

Therefore the remainder is $a^2 - 9ab + 7b^2$.

Since this is an identity, it is true for any values of a and b. Hence, the work may be checked by letting $a = 1, b = 2$. The minuend then becomes 2, and the subtrahend -9, and the remainder 11, which is $2 - (-9)$, as follows:

$$\begin{array}{r} 4 - 10 + 8 = 2 \\ \underline{3 + 8 - 20 = -9} \\ 1 - 18 + 28 = 11 \end{array}$$

45. Consider now the case of $a - (-b)$.

What must be added to $-b$ to make a? Evidently if we add b to $-b$ we get 0; hence we must also add a. Therefore, the number which added to $-b$ makes a, is $a + b$.

$$\begin{array}{r} a \\ \underline{-b} \\ a + b \end{array}$$

Hence, *the subtraction of a negative quantity is the same as the addition of its absolute value.*

This is often expressed by saying that *minus a minus is plus.*

For example,

$$6a - 2a = 4a,$$

or $$6a - (3a - a) = 6a - 3a + a = 6a - 2a = 4a;$$

but $$6a - (-2a) = 6a + 2a = 8a.$$

EXERCISE XIII

From the first of the quantities given in each exercise subtract the second. Check the work as may be required by the teacher.

1. $a^2 + 2\,ab + b^2,\ ab + b^2.$

2. $3\,x^3 + 2\,x^2y,\ 2\,x^3 - 2\,x^2y.$

3. $2\,a - 3\,b + c,\ a - 3\,b - c.$

4. $m^2 - 2\,mn + n^2,\ m^2 + 2\,mn + n^2.$

5. $a^3 + 3\,a^2b + 3\,ab^2,\ a^3 - a^2b + 3\,ab^2.$

6. $ax^2 + 3\,ay^2 - 4\,z^2,\ 2\,ax^2 + 3\,ay^2 - 4\,z^2.$

7. $pq + 3\,qr + 4\,r^2s,\ -pq - 3\,qr + 4\,r^2s.$

8. $5\,abc + 6\,bcd + 7\,cde,\ 4\,abc - 10\,bcd - 8\,cde.$

9. $b^3 + 3\,b^2c + 3\,bc^2 + c^3,\ \ b^3 - 3\,b^2c + 3\,bc^2 - c^3.$

10. $y^6 + x^4y^3 - 7\,x^3y^4 - x^6,\ x^6 - y^6 + 6\,x^3y^4 - 2\,x^4y^3.$

11. $10\,a^2 + 8\,ab + 14\,b^2,\ -5\,a^3 - 4\,a^2 + 10\,ab + 10\,b^2.$

12. $ab^2c - 3\,a^2bc + 4\,abc^2,\ -6\,a^4 + 5\,b^4 - 3\,abc^2 + 4\,a^2bc.$

13. $a^4 + 2\,a^3b + 3\,a^2b^2 - 4\,ab^3 + b^4,\ 2\,a^3b - 3\,a^2b^2 - 5\,ab^3 + b^4.$

14. $ax^3 + bx^2y - cxy^2 - 3\,dy^3,\ 6\,bx^2y + 5\,cxy^2 - 2\,ax^3 + 7\,dy^3.$

15. $a^2b^2 + a^4 + b^4 - 3\,a^3b + 4\,ab^3,\ -a^4 - b^4 - a^2b^2 - 3\,a^3b - 4\,ab^3.$

16. If $P = a^2 + 2\,ab + b^2$, $Q = 2\,a^2 + ab + b^2$, and $R = -4\,ab - 7\,b^2$, find the value of $P + Q - R$.

17. Find the result of subtracting the sum of $2\,a^2 - 14\,ab - 3\,b^2$ and $3\,a^2 + 5\,ab - 4\,b^2$ from the sum of $4\,a^2 - 10\,ab - 3\,b^2$ and $a^2 + ab - 4\,b^2$.

18. What is the difference between the capital of a man who has a stock of goods worth \$5000, \$750 in the bank, and owes \$1000 on a mortgage, and that of one who has a stock of goods worth \$6000, has overdrawn his bank account \$275, and owns a \$500 mortgage?

46. Detached coefficients. Additions and subtractions may evidently be performed without the labor of writing down all of the letters. Since the coefficients of like terms are added, these coefficients may be detached and added separately, the coefficients of like terms being placed under one another. Missing terms are indicated by zeros.

Thus, the second of the following additions is the simpler:

$$\begin{array}{rrr}
(1) & (2) & \text{Check.} \\
a^2 + 2\,ab + b^2 & 1 + 2 + 1 & = 4 \\
-3\,a^2 - ab + b^2 & -3 - 1 + 1 & = -3 \\
4\,a^2 - 3\,ab - 3\,b^2 & 4 - 3 - 3 & = \underline{-2} \\
\hline
2\,a^2 - 2\,ab - b^2 & 2 - 2 - 1 & = -1
\end{array}$$

$$2\,a^2 - 2\,ab - b^2.$$

Since, if the arbitrary value 1 is assigned to each letter, the value of each term is its numerical coefficient, the check requires merely the addition of the coefficients.

EXERCISE XIV

Perform the operations indicated in Exs. 1–6 by using detached coefficients.

1. Add $a^3b + a^2b^2 - 4\,ab^3$, $3\,a^3b - b^4$, $-a^2b^2 + b^4$, $4\,ab^3$.
2. Add $5\,x^4 - 2\,x^2y^2 + y^4$, $x^3y + xy^3$, $x^4 - xy^3$, $-x^3y + y^4$.
3. Add $x^3 - x^2y + xy^2 - y^3$, $2\,x^3 + 3\,x^2y - 4\,xy^2 + y^3$, $x^3 - y^3$.
4. Add $p^3 + 3\,p^2 + 4\,p - 6$, $-p^2 - 2\,p + 1$, $p^3 - 1$, $3\,p^3 + 2\,p + 3$.
5. From $a^2 + 2\,ab + b^2$ subtract $a^2 - 2\,ab + b^2$.
6. From $x^3 + x^2y + xy^2 + y^3$ subtract $x^3 - x^2y + xy^2 - y^3$.

Given $P = x^3 + 3\,x^2y + 3\,xy^2 + y^3$, $Q = -3\,x^2y + 3\,xy^2 - 3\,y^3$, $R = x^3 - y^3$, find by using detached coefficients the values of the expressions in Exs. 7–14, checking as above.

7. $P - Q$. 8. $Q - R$. 9. $R - P$. 10. $Q - P$.
11. $R - Q$. 12. $P - R$. 13. $P + Q + R$. 14. $P + Q - R$.

III. SYMBOLS OF AGGREGATION

47. Symbols of aggregation, preceded by the symbols + and −, may be removed by considering the principles of addition and subtraction already learned.

Since $$a + (b - c) = a + b - c,$$
and $$a - (b - c) = a - b + c, \qquad \S\ 44$$

therefore, *a symbol of aggregation preceded by + may be neglected; if preceded by − it may be removed by changing the sign of each term within.*

E.g., $2a + (3b - c + a) = 2a + 3b - c + a = 3a + 3b - c.$

$$2a - (3b - c + a) = 2a - 3b + c - a = a - 3b + c.$$

For the same reasons, *any terms of a polynomial may be enclosed in a symbol of aggregation preceded by +, without change; also in a symbol of aggregation preceded by −, provided the sign of each term within is changed.*

E.g., $x + 3y - 4z = x + (3y - 4z).$

$$a + b - c + d = a + (b - c + d) = a + b - (c - d).$$

The word **term** now takes on a broader meaning than that given in § 3. *E.g.*, in the expression $a - b(c - d)$, $b(c - d)$ is often considered as a *term*. So in general, where no confusion will arise, polynomials enclosed in symbols of aggregation, either with or without coefficients, are often called *terms*.

E.g., $(a - b)x^2 + (a + b)x + (a^2 - b^2)$ may be considered as a trinomial.

Similarly, $2(a + b - c) + 3(a - b + c)$ may be considered as a binomial.

EXERCISE XV

Remove the symbols of aggregation in the following:

1. $a-(a-b+c)$.
2. $p^2+2pq+q^2-(q^2-p^2)$.
3. $a^2-3b^2+(2a^2+7b^2-c^2)$.
4. $a^3-(3a^2b+a^3-b^3)-b^3+3a^2b$.
5. $2x^2-3xy+y^2-(2x^2+3xy-y^2)$.
6. $5m^3-(3m^3+1)-(4m^4+m^3-3)+(m^3+1)$.

Several symbols of aggregation, one within another, may be removed by keeping in mind the principles mentioned on p. 40.

The order in which these symbols are removed cannot affect the result, but the simplest plan will be discovered by considering the following solution.

Simplify $a-[a+b-(c-\overline{d-e})+c]$, (1) beginning with the inner symbol, (2) beginning with the outer symbol.

(1)	(2)
1. $a-[a+b-(c-\overline{d-e})+c]$	1. $a-[a+b-(c-\overline{d-e})+c]$
2. $=a-[a+b-(c-d+e)+c]$	2. $=a-a-b+(c-\overline{d-e})-c$
3. $=a-[a+b-c+d-e+c]$	3. $=a-a-b+c-\overline{d-e}-c$
4. $=a-a-b+c-d+e-c$	4. $=a-a-b+c-d+e-c$
5. $=-b-d+e.$	5. $=-b-d+e.$

It will be noticed that there are more changes in signs made in solution (1) than in solution (2), with correspondingly greater opportunity for mistakes. Furthermore, in (2) it is an easy matter to write step 5 as soon as step 2 is obtained, while in (1) this is not so easy. Hence the solution (2) is in this case the better.

While cases sometimes arise where it is better to remove the inner parentheses first, especially where there is a coefficient before the parentheses, in general *it is better to remove the outer parentheses first.* A little practice will enable the student to remove them all at sight if this plan is followed.

EXERCISE XVI

Remove the symbols of aggregation in the following expressions, uniting the numbers or the like terms in each result:

1. $13-(7-2)+3.$
2. $13-7-(2+3).$
3. $13-(7-2+3).$
4. $13-(7-2)+3.$
5. $13-[7-(2+3)].$
6. $63-24-(15-8).$
7. $63-(24-15)-8.$
8. $63-(24-15-8).$
9. $63-24-(15-8).$
10. $63-[24-(15-8)].$
11. $a-[a-(b+c)].$
12. $2a+b-(a-2b).$
13. $4x-[2x-(x+y)+y].$
14. $-[a^2-(2ab-\overline{b^2-a^2})+b^2].$
15. $12ax+3b^2+[6ax-(2b^2+7ax)].$
16. $4a^2-\{5b^2+a-[6a^2-3a-(b^2-a)]\}.$
17. $a^2x-[ax^2+a^2-(a^2x-a^2)+x^2]-ax^2+x^2.$
18. $24a^2+3ab-[2ab+6a^2-(a^2-4ab)+a^2].$
19. $10m^2+5mn-[6m^2+n^2-(2mn-\overline{m^2+n^2})]-n^2.$
20. $-(-(-(\cdots-(-1)\cdots)))$, an even number of sets of parentheses; an odd number of sets.

REVIEW EXERCISE XVII

1. Solve the equation $15 - [5 - (2 - \overline{1 - x}) + 13] = 0$.

2. What is the degree of $x^4 + 3x^3 - 4x^2 + 9x - 111$?

3. Solve the equation $3 - (2 - \overline{1 - x}) - [1 - (1 - x)] = 0$.

4. Subtract $a^3 - 3a^2b + 3ab^2 - b^3$ from $a^3 + 3a^2b + 3ab^2 - b^3$.

5. Give two monomials of the same degree; also two like terms.

6. Solve the equation $4x + 44 = 5x - 55$, and prove that the result is correct.

7. What is the test of accuracy in the solution of an equation? Illustrate.

8. What number is that which, if added to 47, equals 19 less than twice the number?

9. What is the sum of $a^2 + 2ab + b^2$, $-a^2 - 2ab + b^2$, $a^2 + 2ab - b^2$, and $-a^2 - 2ab - b^2$?

10. What number is that which, if subtracted from 47, equals 19 more than twice the number?

11. From the sum of $x^2 + 2xy + y^2$ and $x^2 - 2xy + y^2$, subtract the sum of $x^2 - 4xy$ and $4xy + y^2$.

12. In the expression $\frac{1}{2}x^2y^3$, name the coefficient of the literal part, and the exponent of each literal factor.

13. Represent algebraically the sum of the third power of a number, 3 times the fourth power of another number, and 5 times the product of the squares of the numbers.

14. Find a number such that if it is multiplied by 11 and the product is added to 137, the result is the same as if the number had been multiplied by 5 and the product added to 359.

CHAPTER III

MULTIPLICATION

I. DEFINITIONS AND LAWS

48. Multiplication originally had reference to positive integers and was a short form of addition. It was, for this case, defined as the operation of taking a number called the **multiplicand** as many times as an addend as there are units in an abstract number called the **multiplier**, the result being called the **product**. In this limited sense, to multiply $2 by 3 is to take $2 three times, thus, $3 \times \$2 = \$2 + \$2 + \$2 = \$6$.

49. With the advance of mathematics it became necessary to multiply by **simple fractions**, and hence to enlarge the definition to include this case. By the primitive meaning of the word *times* it is impossible to take $2 *two-thirds of a time*. But the product of $2 by $\frac{2}{3}$ may be *defined* as $\frac{2 \times \$2}{3}$.

So the product of c by $\frac{a}{b}$ may be *defined* as the product of a and c, divided by b, c being either integral or fractional.

50. With the further advance of mathematics it became necessary to multiply by **negative numbers**, and hence to enlarge the definition to include this case. The natural definition will appear from a simple illustration.

Suppose 5 men move into a town, each paying $1 a week in taxes. They are worth $5 \times \$1 = \$5$ a week to the town.

Suppose 5 such men move out. This may be represented by saying that the town gains -5 men, or, in money, $-\$5$.

Suppose 5 vagrants move in, each being a charge of \$1 a week. They are worth $5 \times (-\$1) = -\$5$ a week to the town.

Suppose 5 such vagrants move out. This may be represented by saying that the town gains -5 vagrants, or, in money, \$5.

Hence, it is *reasonable* to say that

\$1 multiplied by		$5 =$	\$5,	for the first		case;
\$1	"	"	$-5 = -\$5,$	"	second	"
$-\$1$	"	"	$5 = -\$5,$	"	third	"
$-\$1$	"	"	$-5 = \$5,$	"	fourth	"

51. From such considerations **multiplication by a negative number** is defined as multiplication by the absolute value of the multiplier, the sign of the product being changed.

E.g., allowing the word *times* to indicate multiplication in general,

-2 times 3 means 2×3 with the sign changed;

-2 " -3 " $2 \times (-3)$ " " "

$$\therefore -2 \times 3 = -6,$$

$$-2 \times -3 = -(-6) = +6.$$

52. General definition of multiplication. The above partial definitions may now be put into one general definition:

To multiply a number (the multiplicand) by an abstract number (the multiplier) is to do to the former what is done to unity to obtain the latter.

E.g., consider the meaning of $3 \times \$2$. Since $3 = 1 + 1 + 1$, therefore, $3 \times \$2$ means $\$2 + \$2 + \$2 = \$6$.

Consider also $\frac{2}{3} \times \frac{5}{7}$. Since $\frac{2}{3} = (1 + 1) \div 3$, therefore, $\frac{2}{3} \times \frac{5}{7}$ means $(\frac{5}{7} + \frac{5}{7}) \div 3$, or $\frac{10}{7} \div 3$, or $\frac{10}{21}$.

Consider also $(-2) \times (-3)$. Since $-2 = -(1 + 1)$, therefore, $(-2) \times (-3)$ means $-[(-3) + (-3)]$, or $-(-6)$, or 6.

53. The result of multiplication is called the **product**, and the product of two abstract numbers is called a **multiple** of either.

54. *The expression* $a \cdot 0$ *is defined to mean* 0.

This is the natural definition, because 2×0 must mean $0 + 0$.

55. And since the order of factors can be changed without altering the product, *the product* $0 \cdot a$ *is defined to be the same as* $a \cdot 0$, *or* 0.

56. **The product of three abstract numbers** is defined to be the product of the second and third multiplied by the first. *I.e.*, abc means the product of b and c multiplied by a.

57. The product of four or more abstract numbers may be understood from the above definition for three. *E.g.*, $abcd$ means cd multiplied by b, and that product by a.

58. **Law of signs.** From the definition it appears that *like signs produce plus, and unlike signs minus.*

I.e.,

$$\begin{array}{ccccc} + & \times & + & = & + \\ + & \times & - & = & - \\ - & \times & + & = & - \\ - & \times & - & = & + \end{array}$$

59. **Reading of products.** As already stated, the original meaning of the word *times* referred to positive integers.

The expressions $\frac{2}{3}$ *times*, $\frac{2}{3}$ *of a time*, and -2 *times* are meaningless in the original sense of the word. But with the extension of the definition of multiplication has come an extension of the meaning of the word *times*, so that it is now generally used for all products, as in § 51.

Thus, the expression $2\frac{1}{2}$ *times as much* is generally used, although it is impossible to pick up a book $2\frac{1}{2}$ times. So $(-2) \times (-3)$ is read "minus 2 times minus 3," although we cannot look out of a window -2 times.

As already stated, the word *into* is sometimes used in algebra to indicate the product of two or more factors, $(-a)(-b)$ being read "$-a$ into $-b$."

The parentheses about negative factors are omitted when no misunderstanding is probable. Thus, $(-a)(-b)$ may be written $-a \times -b$, or even $-a \cdot -b$. But $-a^2$ and $(-a)^2$ are not the same, the former meaning $-aa$ and the latter $-a \cdot -a$, or $+a^2$.

EXERCISE XVIII

Perform the multiplications indicated in Exs. 1–17.

1. $-2 \cdot -7$.
2. $-4 \cdot -\frac{3}{4}$.
3. $2 \cdot a \cdot 4 \cdot 0$.
4. $72 \cdot -\frac{1}{9} \cdot -\frac{3}{4}$.
5. $-\frac{1}{2} \cdot -\frac{1}{3} \cdot -\frac{1}{4}$.
6. $(-2)^2 \cdot (-3)^2$.
7. $(-1)^{17} \cdot (-2)^4$.
8. $4 \cdot 5 \cdot -3 \cdot 2 \cdot \frac{1}{3} \cdot \frac{1}{5}$.
9. $-1 \cdot -2 \cdot -3 \cdot -4$.
10. $1 \cdot -2 \cdot 3 \cdot -4 \cdot 5 \cdot -6$.
11. $1 \cdot (-2)^2 \cdot 3^3 \cdot (-4)^4$.
12. $1 \cdot -2 \cdot 3 \cdot -2 \cdot 5 \cdot -12$.
13. $5 \cdot 3 \cdot 1 \cdot -1 \cdot -3 \cdot -5$.
14. $(-1)^{100} \cdot (-1)^{99} \cdot (-2)^5$.
15. $-a \cdot -b \cdot -c \cdot 1 \cdot 2 \cdot 3$.
16. $-2 \cdot -4 \cdot -6 \cdot -8 \cdot -10$.
17. $4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 \cdot -1 \cdot -2 \cdot -3 \cdot -4$.

Simplify the expressions in Exs. 18–22.

18. $10 - 2[9 - 8(7 - 6)] - 5] - 4.$

19. $-2\{-2[-2(-2) + 2] + 2\}.$

20. $3^2 - [3 - (3^2 - \overline{3 - 1}) - 3 \cdot - 3].$

21. $-2[-2(-2 \cdot \overline{-2 \cdot 3 - 1} + 1) + 1].$

22. $2 + 3 - [1 - 2 \cdot - 3 - (5 + 3 \cdot - 1)].$

Perform the following operations:

23. $32 \cdot (-1) - 32 \cdot 2.$

24. $22 - 2 \cdot (-11) \cdot (-1).$

25. $21 \cdot (-3) + 3 \cdot 21 + 5 \cdot 6.$

26. $2 \cdot (-3) \cdot (-4) + 4 \cdot 3 \cdot 2.$

27. $15 \cdot (-2) \cdot (-3) + 4 \cdot (-5) \cdot 6.$

28. $1 \cdot 2 \cdot 3 + 3 \cdot 4 \cdot 5 + (-5) \cdot (-6) \cdot (-7).$

29. $2 \cdot (-3) \cdot 4 + 4 \cdot (-3) \cdot 5 + 3 \cdot (-3) \cdot 4.$

30. $(-2) \cdot (-3) + (-4) \cdot (-5) + (-5) \cdot (-6).$

31. $(-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot (-5) + 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5.$

60. The index law. Since $a^2 = aa$, and $a^3 = aaa$, therefore $a^2 \cdot a^3 = aa \cdot aaa = a^5$. Similarly, if m and n are positive integers,

$$a^m = aaa \cdots \text{ to } m \text{ factors,}$$

and

$$a^n = aaa \cdots \text{ to } n \text{ “}$$

$$\therefore\ a^m \cdot a^n = aaaa \cdots \text{ to } m + n \text{ “}$$

$$\therefore\ a^m \cdot a^n = a^{m+n}.$$

This is known as the **index law of multiplication.**

Hence, $2a^2b^3c^5 \cdot 5a^3b^2c^4 = 10a^5b^5c^9$.

The cases in which m and n are negative, zero, or fractional are considered later.

EXERCISE XIX

Perform the multiplications indicated.

1. $a^2 \cdot 2\,ab \cdot b^2$.
2. $-a^2 \cdot (-a)^2$.
3. $x^m y^n \cdot x^n y^m \cdot x^2 y^2$.
4. $a^m b^n \cdot a^n b^m \cdot a^2 b^2$.
5. $x^2 y^3 z^2 \cdot x^4 y z^3 \cdot x y^6 z$.
6. $a x^2 y z \cdot b x y^2 z \cdot c x y z^2$.
7. $25\,ab^2c^3d^4 \cdot 2\,a^4b^3c^2d$.
8. $5\,m^2n \cdot 6\,mn^2 \cdot 3\,mn$.
9. $a^{10}x^9 \cdot a^8x^5 \cdot a^6x^3 \cdot a^4x$.
10. $-a \cdot -a^2 \cdot -a^3 \cdot -a^4 \cdot -a^5$.
11. $ax^5 \cdot bx^5 \cdot cx^5$.
12. $abc^2 \cdot ab^2c \cdot a^2bc$.
13. $amx^2y \cdot bnx^2y \cdot cpx^2y$.
14. $m^2n^3 \cdot m^3n^4 \cdot m^4n^5 \cdot m^3$.
15. $-a^2x^3 \cdot -a^3x^4 \cdot -a^4x^5$.
16. $a^3b^2c \cdot a^4b^3c^2 \cdot a^5b^6c^7 \cdot bc^2$.
17. $-a^2mx^3 \cdot -am^2x \cdot am$.
18. $x^2y^3z^4 \cdot xy^2z^3 \cdot x^6y^4z^2 \cdot xyz$.
19. $x^5y^2z \cdot x^2y^5z \cdot xy^5z^2 \cdot xy^2z^5$.
20. $3\,am^2x \cdot 4\,a^2mx^2 \cdot 5\,a^3mx^3$.
21. $p^2q \cdot -pq^2 \cdot -p^3q^3 \cdot -pq$.
22. $wx^2yz \cdot w^2xyz \cdot wx^4yz^3 \cdot wy$.
23. $am^2x \cdot bn^2x \cdot ab \cdot a^2b^2m^2n^2x^2$.
24. $pqr \cdot -pqr \cdot -p^2q^2r^2 \cdot -p^3q^3r^3$.
25. $x^7 \cdot x^6 \cdot x^5 \cdot y^5 \cdot y^6 \cdot y^7 \cdot z^9 \cdot z^8 \cdot z$.
26. $p^2q \cdot (-2\,pq^2) \cdot (-3\,p^4q^4) \cdot (-4\,pq)$.
27. $-a \cdot (-a)^2 \cdot (-a)^3 \cdot (-a)^4 \cdot (-a)^5$.
28. $ab^2c^3d^4e^5 \cdot (-a^5b^4c^3d^2e) \cdot (-a^2b^2c^2d^2e^2)$.
29. $-3\,ab^2x \cdot (-3\,a^2bxy) \cdot (-3\,aby^2) \cdot (-xy)$.

II. MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

61. I. *When the monomial is a positive integer*, as in the case of $a(b-c)$.

First, to take a simple case, what does $2(x-y)$ mean? Evidently that $x-y$ is taken twice; that is,

$$2(x-y) \text{ means } x-y+x-y=2x-2y.$$

And, in general, what does $a(b-c)$ mean? Evidently that $b-c$ is taken a times; that is, that b is taken a times and $-c$ is taken a times.

$$\therefore\ a(b-c)=ab-ac.$$

II. *When the monomial is a negative integer*, as in the case of $-m\cdot(b-c)$.

Since multiplying by a negative multiplier is the same as multiplying by the absolute value of the multiplier, the sign of the product being changed (§ 51),

$$\begin{aligned}\therefore\ -m\cdot(b-c)&=-(mb-mc)\\&=-mb+mc.\end{aligned}$$

62. From the results of these two cases it appears that:

To multiply a polynomial by an integral monomial is to multiply each term of the polynomial by the monomial and to add the products.

E.g., $\quad -3x(x^2-xy+y^2)=-3x^3+3x^2y-3xy^2.$

These results may be checked by putting any arbitrary values in place of the letters. Thus let

$$x=2 \text{ and } y=1;$$

then $\quad -3\cdot 2(4-2+1)=-3\cdot 8+3\cdot 4-3\cdot 2,$

or $\quad -18=-18.$

EXERCISE XX

Perform the following multiplications:

1. $a^2(a^2+b^2-c^2)$.
2. $2ab(ab-bc-ac)$.
3. $5a^2(5+6a^2-7a^4)$.
4. $-5a^m(a^m+a^n+1)$.
5. $-3xyz(x^2-y^2-z^2)$.
6. $-7a^2b(13bc^2-9b^2c)$.
7. $5m^3xy(xz^2-3z^2x-4)$.
8. $9a^2bc(2ab^2c^2-4a^5b^6c^6)$.
9. $-7x^2y(-3xy^2+2xy)$.
10. $25ab^2c^3d^4(2x^4b^3+2c^2d)$.
11. $ax^2(a^2x+a+x^2)$.
12. $mnp^2(m+n+p^2)$.
13. $17a^2(2a^2-3b^2-5a^2c)$.
14. $-2a^2(-2a^2-2a-2)$.
15. $-5a[-3a+2(a-2)]$.
16. $-3a^3b^5c^7(a^8b^6c^4-13a^4b^2)$.
17. $-15x^7y^8(m^7n^8x^2y^2-2p^9q^{10})$.
18. $24a^3b^2(2a^2b^3-3ab^2+a^7b^{11})$.
19. $-5p^4q^3(m^5n^5pq^2+6x^7y^8p^5q^7)$.
20. $15ab^2cx(5acx+6bcx+2x^2)$.
21. $2a^2bc(3ab^2c+6abc^2-5abc)$.
22. $5c^4d^4(8cd^2+34c^4d^6-24c^6d^7)$.
23. $5m^2n^3(2m^3n^2p^5q^5+3m^5n^5x^9y^{10})$.
24. $2p^2qr(pq^2r+3pqr^2+pr-qr)$.
25. $amp^2q(a^2mpq+am^2pq+ampq^2)$.
26. $13a^2y^2(8a^5y^7-2a^4y^9+a^5y^9-y^2)$.
27. $-7m^2n^3(2m-3n-4mn+6m^3n)$.
28. $-2ab^2c^3(-3a^3b^2c-5ab^3c^2+6abc)$.
29. $-abcd(-2a^2bcd-3ab^2cd-4abc^2d-5abcd^2)$.

III. MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

63. Suppose, for example, it is required to multiply $c+d$ by $a+b$.

Let $a+b$ be designated by m. Then $(a+b)(c+d)$ becomes $m(c+d)$.

But $$m(c+d)=mc+md, \qquad \text{by § 61.}$$

And because $m=(a+b)$, this becomes

$$(a+b)c+(a+b)d.$$

But by § 61, this equals $ac+bc+ad+bd$.

From this it appears that *to multiply one polynomial by another is to multiply each term of the first by each term of the second and to add the products.*

The following example illustrates the process:

$$\begin{array}{ll}
 & x^2+2xy+y^2 \\
 & \qquad\quad x+y \\
\hline
\text{Product by } x, & x^3+2x^2y+xy^2 \\
\text{Product by } y, & \qquad\quad x^2y+2xy^2+y^3 \\
\hline
\text{Sum of products,} & x^3+3x^2y+3xy^2+y^3
\end{array}$$

To check this work we may let $x=1$, $y=1$. Then

$$1+2+1=4$$
$$1+1=2$$
$$1+3+3+1=8, \text{ or } 2\cdot 4.$$

EXERCISE XXI

Perform the following multiplications:

1. $(a+b)(x+y)$.
2. $(x+y)(x-y)$.
3. $(x^2-xy+y^2)(x+y)$.
4. $(p^2+pq+q^2)(p-q)$.
5. $(p+q)(p^2-pq+q^2)$.
6. $(2a-4b)(a^2-ab+b^2)$.

7. $(x^2 + x - 4)(x^2 - x + 4)$.

8. $(x^2 + 3xy + 2y^2)(x - y)$.

9. $(a^3 + a^2 + a + 1)(a - 1)$.

10. $(x^2 + 2x + 3)(x^2 - 2x + 3)$.

11. $(6x^2 + 2x + 1)(x^2 - x - 1)$.

12. $(5x^2 - x + 1)(3x^2 - x - 2)$.

13. $(a^2 + ab + b^2)(a^2 - ab + b^2)$.

14. $(2x + 3y - z)(2x - 3y + z)$.

15. $(x^4 + x^3 + x^2 + x + 1)(x - 1)$.

16. $(x^2 + 4xy + y^2)(x^2 + xy - y^2)$.

17. $(x^2 - 2xy + y^2)(x^2 + 2xy + y^2)$.

18. $(a^2 - 3ab + b^2)(a^2 - 3ab - b^2)$.

19. $(x + y)(x^3 + 3x^2y + 3xy^2 + y^3)$.

20. $(x^2 + x + 1)(x^3 - 2x^2 + 3x - 2)$.

21. $(c^2 + 3cd + 4d^2)(c^2 - 3cd - 4d^2)$.

22. $(3a^2 - 2a)(5a^3 - 2a^2 - 3a + 4)$.

23. $(x^3 + x^2 + x + 1)(x^3 - x^2 + x - 1)$.

24. $(m^2 + mn + 2n^2)(m^2 + mn - 2n^2)$.

25. $(5x + 1)(125x^3 - 25x^2 + 5x - 1)$.

26. $(2p^2 - 3pq + q^2)(2p^2 + 3pq + q^2)$.

27. $(a^2 - 4ab - b^2)(a^3 + a^2b + ab^2 + b^3)$.

28. $(p^2 + 2pq + q^2)(p^3 + 3p^2q + 3pq^2 + q^3)$.

29. $(a^3 - 2a^2 + 2a - 1)(a^3 + 2a^2 + 2a + 1)$.

30. $(a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$.

64. A polynomial is said to be **arranged according to the powers of some letter** when the exponents of that letter in the successive terms either increase or decrease continually.

In the former case the polynomial is said to be *arranged according to ascending powers*, in the latter *according to descending powers* of the letter.

E.g., $x^5 + 3x^3 + x^2 + 1$ is arranged according to descending powers of x.

The polynomial $x^3 - 3x^2y + 3xy^2 - y^3$ is arranged according to descending powers of x and ascending powers of y.

There is evidently an advantage in arranging both multiplicand and multiplier according to the powers of some letter, as shown by the following example:

NOT ARRANGED

$$\begin{array}{l} y^2 + x^2 + 2xy \\ \qquad x + y \\ \hline xy^2 + x^3 + 2x^2y \\ \qquad\qquad\qquad + y^3 + x^2y + 2xy^2 \\ \hline xy^2 + x^3 + 2x^2y + y^3 + x^2y + 2xy^2 \\ \qquad = x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

ARRANGED

$$\begin{array}{l} x^2 + 2xy + y^2 \\ \qquad\qquad x + y \\ \hline x^3 + 2x^2y + xy^2 \\ \qquad\quad x^2y + 2xy^2 + y^3 \\ \hline x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

Check. Let $x = 1$, $y = 1$. Then $2 \cdot 4 = 8$.

The method at the right is evidently much simpler.

65. There are certain elementary laws, the names of which should be familiar to students, but the proof of which may be left for more advanced courses. These fundamental laws of algebra are as follows:

1. **The associative laws,** in which certain quantities are associated together.

In addition: $a + b + c = (a + b) + c$, or $a + (b + c)$.

In multiplication: $abc = (ab)c$, or $a(bc)$.

2. **The commutative laws,** in which the order of certain quantities is changed.

In addition: $a + b = b + a.$

In multiplication: $ab = ba.$

3. **The distributive law,** that $m(a + b) = ma + mb$, as proved on p. 50.

EXERCISE XXII

Perform the following multiplications:

1. $x^3 - y^3$ by $x^3 + y^3$.
2. $a^2x + x^2a$ by $x^2a - a^2x$.
3. $x^2y^2 - x^3 - y^3$ by $y - x$.
4. $x + y + z$ by $x + y - z$.
5. $x^4 - x^2y^2 + y^4$ by $x^2 + y^2$.
6. $x^2 - 2xy + y^2$ by $x^2 - y^2$.
7. $1 - a^2 + a^4 - a^6$ by $1 + a^2$.
8. $a^5 + b^5 + c^5 - abc$ by $a - b$.
9. $x^2 + 3xy - 4y^2$ by $x^2 - 8y^2$.
10. $x^7 + x^5y + y^6$ by $x^2 - 3x + y$.
11. $x^4 + 2x - 3$ by $x^3 - 2x^2 + 3$.
12. $x^3 + 2x - 2$ by $x^3 + 2x^2 + 2$.
13. $a^2x^5 + a^5x^2 + a^7 + x^7$ by $a + x$.
14. $x^5 - x^3 + 1$ by $x^5 + x^4 - x^2 + 1$.
15. $xyz - x^2 - y^2 - z^2$ by $x + y + z$.
16. $p^2 - 3pq + q^2$ by $p^2 + 3pq + q^2$.
17. $m^3 + 3m - 7$ by $m^3 - 3m^2 + 1$.
18. $p^2 - 2pq + q^2$ by $p^2 + 2pq + q^2$.

19. $p^8 - 2p^4 + p^2 - 3$ by $p^4 - p^2 + 1$.

20. $a^4 - 2a^2b^2 + ab^3$ by $a^4 - a^3b + b^4$.

21. $a^m + a^nb^n + b^n$ by $a^m - a^nb^n - b^n$.

22. $a^5 + a^4 + a^3 + a^2 + a + 1$ by $a - 1$.

23. $m^2 - 4mn - n^2$ by $m^2 + 4mn + n^2$.

24. $-a^2 + 3ab + b^2$ by $3ab - b^2 + a^2$.

25. $a^5 - a^4 + a^3 - a^2 + a - 1$ by $a + 1$.

26. $a^4x^4 - a^3x^3 + a^2x^2 - ax + 1$ by $ax + 1$.

27. $x^6 + x^5y - x^3y^3 + y^6$ by $x^2 - xy - 2y^2$.

28. $2a^2 - 3ab + 4b^2$ by $4a^2 - 3ab + 2b^2$.

29. $a^7 - a^5b^2 + a^3b^4 - b^7$ by $a^6b + a^4b^3 - ab^6$.

30. $a^2b^3 - ab^4 + b^5$ by $a^5 - a^4b + a^3b^2 - a^2b^3$.

31. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.

32. $-a^3 + a^2b - ab^2 + b^3$ by $-a^2 + ab - b^2$.

33. $x^3 - 2x^2y + 3xy^2 - y^3$ by $x^2 - 2xy - 4y^2$.

34. $p^6 + p^5q - pq^5 + q^6$ by $p^4q^2 + p^3q^3 - 2p^2q^4$.

35. $x^3 - 3x^2y + 4xy^2 + 5y^3$ by $x^3 - 2xy^2 + 2y^3$.

36. $xy + 2xz - 3yz + x^2 + y^2 + 4z^2$ by $x - y - 2z$.

37. $x^4 - x^3y + 2x^2y^2 - 3xy^3 + 2y^4$ by $x^2 - 3xy + 2y^2$.

66. **Detached coefficients** may be employed in multiplication whenever it is apparent what the literal part of the product will be.

E.g., in multiplying $x^7 + pz + q$ by $x^6 - x + pq$ the coefficients cannot be detached to advantage.

But in multiplying $x^2 + 2xy + y^2$ by $x + 3y$, it is apparent that the exponents of x decrease by 1 while those of y increase by 1 in each factor, and that this law will also hold

in the product. Hence, when the coefficients are known the product is known also, and the multiplication may be performed as follows:

$$\begin{array}{rl} & \textit{Check.} \\ 1+2+1 & =4 \\ \underline{1+3} & =4 \\ 1+2+1 & \\ \underline{3+6+3} & \\ 1+5+7+3 & =16 \end{array}$$

$\therefore\ (x+3y)(x^2+2xy+y^2)=x^3+5x^2y+7xy^2+3y^3.$

67. In case any powers are lacking in the arrangement of the polynomial, zeros should be inserted to represent the coefficients of the missing terms.

E.g., to multiply x^2+xy+y^2 by x^2+y^2, the following arrangement may be used:

$$\begin{array}{rl} 1+1+1 & \qquad \textit{Check.}\quad 2\cdot 3=6. \\ \underline{1+0+1} & \\ 1+1+1 & \\ \underline{1+1+1} & \\ 1+1+2+1+1 & \qquad x^4+x^3y+2x^2y^2+xy^3+y^4 \end{array}$$

EXERCISE XXIII

Perform the multiplications indicated in Exs. 1–8 by detached coefficients.

$P=x^3-x^2y+xy^2-y^3,\ Q=x-y,\ R=x^2-xy+y^2.$

1. PQ. 2. PR. 3. QR. 4. P^2.

5. Q^2R. 6. R^2. 7. QR^2. 8. Q^2R^2.

Expand the following, using detached coefficients:

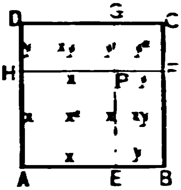
9. $(x+y)^2$. 10. $(x-y)^2$. 11. $(x+y)^3$.

12. $(x-y)^3$. 13. $(x+y)(x-y)$.

IV. SPECIAL PRODUCTS FREQUENTLY MET

68. In Exs. 9–13 on p. 47 five products were found which are so frequently used as to require memorizing. They are as follows:

1. $(x+y)^2=x^2+2xy+y^2$. Hence, ***the square of the sum of two numbers equals the sum of their squares plus twice their product.***



This may be illustrated by a figure. Here $ABCD$ is the square on $x+y$, and it evidently equals the sum of the square on x, the square on y, and the two rectangles whose areas are $x\times y$.

$$\therefore (x+y)^2=x^2+2xy+y^2.$$

2. $(x-y)^2=x^2-2xy+y^2$. Hence, *the square of the difference of two numbers equals the sum of their squares minus twice their product.*

3. $(x+y)(x-y)=x^2-y^2$. Expressions of the form $x+y$, $x-y$, are called *conjugates* of each other. Hence, *the product of two conjugate binomials equals the difference of the squares of the two terms* (*i.e.*, the square of the minuend minus the square of the subtrahend).

4. $(x+y)^3=x^3+3x^2y+3xy^2+y^3$. Hence, *the cube of the sum of two numbers equals the sum of their cubes, plus three times the second into the square of the first, plus three times the first into the square of the second.*

5. $(x-y)^3=x^3-3x^2y+3xy^2-y^3$. Hence, *the cube of the difference of two numbers equals the cube of the first, minus three times the second into the square of the first, plus three times the first into the square of the second, minus the cube of the second.*

EXERCISE XXIV

By the help of the theorems of § 68 expand the following expressions:

1. 42^2.
2. $(a^2 - 3b)^2$.
3. 49^2.
4. $(2x + 3y)^2$.
5. $49 \cdot 51$.
6. $(3x - 2y)^2$.
7. 23×17.
8. $(10x - 7y)^2$.
9. 95×85.
10. $(2m - 3n)^2$.
11. $(a^2 + 3)^2$.
12. $(8x + 11y)^2$.
13. $(a^7 - 2)^2$.
14. $[a - (b + c)]^2$.
15. $(x^3 + 3)^2$.
16. $(a^2 + 3)(a^2 - 3)$.
17. $(x^5 - 5)^2$.
18. $(x^4 + y^4)(x^4 - y^4)$.
19. $(x^4 + 4)^2$.
20. $(ax + 2)(ax - 2)$.
21. $(x^3 - 5)^2$.
22. $[(a + b)(a - b)]^2$.
23. $(2p + 1)^2$.
24. $(2x^2 + 1)(2x^2 - 1)$.
25. $(5a - 2)^2$.
26. $(a^2 + 2b)(a^2 - 2b)$.
27. $(2x^3 - 1)^2$.
28. $(abc^2 + 7)(abc^2 - 7)$.
29. $(2x^2 - y)^2$.
30. $(m^3n^2p - q)(q + m^3n^2p)$.
31. $(x - y - xy)(x - y + xy)$.
32. $(ax + b + 1)(ax + b - 1)$.
33. $(a + b - ab)(a + b + ab)$.
34. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.
35. $(a^2 + x^2 + xy)(a^2 + x^2 - xy)$.
36. $(m + n + mn)(m + n - mn)$.
37. $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$.
38. $(x^2 + 4x + 4)(x^2 - 4x + 4)$.

CHAPTER IV

DIVISION

I. DEFINITIONS AND LAWS

69. **Division** is the operation by which, having the product of two expressions and one of them (not zero) given, the other is found.

Thus, 6 is the product of 2 and 3; given 6 and 2, 3 can be found. But 0 is the product of *any* finite number and 0; hence, in this case, the other factor cannot be found.

The given product is called the **dividend**, the given expression is called the **divisor**, and the required expression is called the **quotient**.

70. Since $0 = a \cdot 0$ (§ 54), it follows that $\frac{0}{a}$ should be defined to mean 0.

71. Law of signs. Since

$$+a \cdot +b = +ab,$$
$$+a \cdot -b = -ab,$$
$$-a \cdot +b = -ab,$$

and $$-a \cdot -b = +ab;$$

therefore $$+ab \div +a = +b,$$
$$-ab \div +a = -b,$$
$$-ab \div -a = +b,$$

and $$+ab \div -a = -b.$$

That is, *like signs in dividend and divisor produce* $+$, *and unlike signs* $-$, *in the quotient.*

72. Index law. Since $a^{m-n} \cdot a^n = a^m$, by the index law of multiplication (§ 60), therefore, $\frac{a^m}{a^n} = a^{m-n}$, by the definition of division.

Hence, $10\,a^5b^5c^9 \div 5\,a^3b^2c^4 = 2\,a^2b^3c^5$.

The above proof is based on the supposition that $m > n$, and that both are positive integers. The cases in which m and n are zero, negative, and fractional, and in which $m < n$, are considered later.

EXERCISE XXV

Perform the following divisions:

1. $-125 \div -25$.
2. $80 \div -16$.
3. $\frac{378}{-63}$.
4. $\frac{3\,a^2bc}{-bc}$.
5. $\frac{252\,ax^2}{36\,ax}$.
6. $\frac{25\,a^7bc^6}{-5\,a^5bc}$.
7. $\frac{8\,w^3x^9y^8z^5}{8\,w^2x^7y^5z}$.
8. $\frac{63\,p^3q^5r^7s}{7\,p^2q^4r^7}$.
9. $\frac{91\,a^5b^5mc^5}{13\,ab^2mc^2}$.
10. $\frac{-10\,xy^2z^8}{-5\,y^2z^3}$.
11. $\frac{84\,a^{10}b^5c^5d^2}{7\,a^6b^2c^3d}$.
12. $\frac{49\,x^{13}y^{20}wz^8}{-7\,wz^2}$.
13. $\frac{-70\,x^2y^3z^4}{7\,xy^2z^3}$.
14. $\frac{-56\,a^{17}b^{14}c}{-8\,a^{16}bc}$.
15. $\frac{75\,ab^2cdp^3q^4}{15\,abpq}$.
16. $\frac{51\,a^3b^2c^3d^4}{-17\,a^2bc^2d}$.
17. $\frac{99\,p^{10}q^{14}r^{15}s^8}{11\,p^2q^7r^9s^3}$.
18. $\frac{-27\,(a-b)}{a-b}$.
19. $\frac{-35\,p^2q^{10}r^{12}}{-7\,p^2qr^{11}}$.
20. $\frac{52\,a^6b^5c^9d^{15}e^{10}}{-13\,a^4bc^3d^7}$.

II. DIVISION OF A POLYNOMIAL BY A MONOMIAL

73. Consider the case of $(ma + mb + mc) \div m$.

Since it has been shown (§ 61) that

$$ma + mb + mc = m(a + b + c),$$

it follows that

$$\frac{ma + mb + mc}{m} = a + b + c,$$

by the definition of division (§ 69).

Hence, *to divide a polynomial by a monomial is to divide each term of the polynomial by the monomial and to add the quotients.*

Thus, $$\frac{x^3 - 2x^2y + xy^2}{-x} = -x^2 + 2xy - y^2.$$

We may check as usual by assigning arbitrary values.

Thus, let $x = 1,\ y = 2;$

then $$\frac{1 - 4 + 4}{-1} = -1 + 4 - 4 = -1.$$

EXERCISE XXVI

Perform the divisions indicated.

1. $\dfrac{15x^2 - 30xy}{5x}$.

2. $\dfrac{21a^2b - 7ab^2}{-7ab}$.

3. $\dfrac{27x^4y - 27xy^4}{-27xy}$.

4. $\dfrac{39p^2q^2 - 13p^3q^3}{3pq}$.

5. $\dfrac{18mn^4 - 27m^4n}{-9mn}$.

6. $\dfrac{24axy^3 + 36a^2xy}{12axy}$.

7. $\dfrac{121m^2n^3 - 110m^3n^2}{-11m^2n^2}$.

8. $\dfrac{x^4 + 3x^3y + 3x^2y^2 + xy^3}{x}$.

9. $\dfrac{-3a^5b - 12a^4b^2 + 9a^3b^3}{-3a^3b}$.

10. $\dfrac{a^3b^2c + 5ab^4c^2 - 2a^2b^2c^2}{ab^2c}$.

11. $\dfrac{a^4 - 3a^3b + 3a^2b^2 - 7ab^3}{-a}$.

12. $\dfrac{200x^3y^5 - 75x^5y^3 + 125x^3y^8}{25x^3y^3}$.

13. $\dfrac{34a^2b^3c - 17ab^2c^3 + 51a^3b^2c}{17ab^2c}$.

14. $\dfrac{65x^{2m}y^2 - 52x^{3m}y^3 + 39x^{4m}y^4}{-13x^my^2}$.

15. $\dfrac{65a^3b^5c^3x^8y^7z^3 - 78a^3b^5c^3x^8y^7z^3}{13a^2b^3x^7y^5}$.

16. $\dfrac{5p^4 - 15p^3q + 10p^2q^3 - 20p^5}{5p^2}$.

17. $\dfrac{2(a+b)^5 - 3(a+b)^3 + 2(a+b)^2}{(a+b)^2}$.

18. $\dfrac{21a^3b^5c^3w^5x^5y^5z^5 - 231a^5b^5cw^8x^6y^4z^2}{21a^2b^3w^4x^3y^2z}$.

19. $\dfrac{48x^{17}y - 36x^{16}y + 72xy^{16} - 108xy}{12xy}$.

20. $\dfrac{(x^2 + 2xy + y^2)^3 + (x^2 + 2xy + y^2)^2}{-(x^2 + 2xy + y^2)}$.

21. $\dfrac{(2x-1)^7 + 5(2x-1)^5 - (2x-1)^3}{-(2x-1)^2}$.

22. $\dfrac{186a^8b^3c^6d^4 - 217a^9b^4c^7d^5 - 155a^{10}b^5c^8d^6}{31a^7bc^3}$.

23. $\dfrac{-52a^{20}b^6 - 78a^6b^{20} - 26a^{12}b^{12} - 130a^6b^6}{-26a^6b^6}$.

24. $\dfrac{-75m^3n^9p^6q^8 - 45m^5n^{11}p^8q^{10} - 60m^7n^{13}p^{10}q^{12}}{-15m^2n^7p^3q^4}$.

25. $\dfrac{-102x^5y^9z^6 - 68x^6y^{10}z^7 - 85x^7y^{11}z^8 + 51x^8y^{12}z^9}{-17x^4y^7z^3}$.

III. DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

74. The process is best understood by considering an example. Let it be required to divide

$$3x^2y + y^3 + x^3 + 3xy^2 \text{ by } y + x.$$

It is readily seen that, if the expressions are rearranged according to the descending powers of x, the first term of the quotient is x^2.

$$\begin{array}{llrl}
 & & x^2 + 2xy + y^2 & = \text{quotient.} \\
\text{Divisor} & = x + y) & x^3 + 3x^2y + 3xy^2 + y^3 & = \text{dividend.} \\
\text{If} & x^2(x+y) \text{ or} & \underline{x^3 + x^2y \qquad\qquad} & \text{is subtracted,} \\
\text{the remainder} & & 2x^2y + 3xy^2 + y^3 & \text{is a new dividend, the product of } x + y \text{ by the rest of the quotient.} \\
 & & & \therefore \text{ the next term of the quotient is } 2xy. \\
\text{Subtracting } 2xy(x+y) \text{ or} & & \underline{2x^2y + 2xy^2 \quad} & \\
\text{the remainder} & & xy^2 + y^3 & \text{is also a new dividend, the product of } x + y \text{ by the rest of the quotient.} \\
 & & & \therefore \text{ the next term of the quotient is } y^2. \\
\text{Subtracting} & y^2(x+y) \text{ or} & \underline{xy^2 + y^3} &
\end{array}$$

there is no remainder, and the division is complete.

75. Exact division. If one of the new dividends becomes 0, the division is said to be *exact*. If not, the degree of some partial dividend will be less than that of the divisor; such a partial dividend is called the **remainder**.

If $D =$ dividend, $d =$ divisor, $q =$ quotient, and $r =$ remainder, then

$$D - r = dq;$$

that is, if the remainder were subtracted from the dividend the result would be the product of the quotient and the divisor.

76. *Checks.* 1. Since the dividend is the product of the quotient and the divisor, one check is by multiplication.

$\because D - r = qd$, any remainder should first be subtracted.

2. The work may also be checked by arbitrary values, but care must be taken that the divisor does not become 0. (§ 69)

77. It is better in practice to abridge the work as in the following example:

$$
\begin{array}{l}
x^2 + 2xy + 2y^2 \\
x+y)x^3 + 3x^2y + 4xy^2 + 5y^3 \\
\underline{x^3 + x^2y} \\
2x^2y \\
\underline{2x^2y + 2xy^2} \\
2xy^2 \\
\underline{2xy^2 + 2y^3} \\
3y^3
\end{array}
$$

It is still better to detach the coefficients if the problem is a simple one.

$$
\begin{array}{l}
1 + 2 + 2 \\
1+1)1 + 3 + 4 + 5 \\
\underline{1 + 1} \\
2 \\
\underline{2 + 2} \\
2 \\
\underline{2 + 2} \\
3
\end{array}
$$

Check. Let $x = y = 1$.

$(1 + 1)(1 + 2 + 2) = 1 + 3 + 4 + 5 - 3$

or $2 \cdot 5 = 10$.

$x^2 + 2xy + 2y^2$, and $3y^3$ remainder.

Similarly, to divide $x^3 - 1$ by $x - 1$.

$$
\begin{array}{l}
1 + 1 + 1 \\
1-1)1 + 0 + 0 - 1 \\
\underline{1 - 1} \\
1 \\
\underline{1 - 1} \\
1 \\
\underline{1 - 1}
\end{array}
$$

Check. Let $x = 2$.

$(2 - 1)(8 - 1) = 4 + 2 + 1$

or $1 \cdot 7 = 7$.

$x^2 + x + 1$.

EXERCISE XXVII

Perform the divisions indicated.

1. $x^3 - y^3$ by $x - y$.
2. $x^{12} - a^{12}$ by $x^3 - a^3$.
3. $32a^5 - b^5$ by $2a - b$.
4. $4x^2 - 9y^2$ by $2x + 3y$.
5. $121x^2 - 1$ by $11x + 1$.
6. $x^2 - 8x + 15$ by $x - 5$.
7. $x^2 - 10x + 24$ by $x - 6$.
8. $4x^2 - 4x + 1$ by $2x - 1$.
9. $16x^4 - 81y^4$ by $2x + 3y$.
10. $16a^4 - 81b^4$ by $2a - 3b$.
11. $16a^2 - 25b^2$ by $4a - 5b$.
12. $x^5 - 5x^2 - 3000$ by $x - 5$.
13. $125x^3 + 27y^3$ by $5x + 3y$.
14. $49m^4 - 81y^2$ by $7m^2 - 9y$.
15. $3x^3 - 7x - 2 - 2x^2$ by $1 + x$.
16. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
17. $a^4 + 24a + 55$ by $a^2 - 4a + 11$.
18. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
19. $x^4 - 2a^2x^2 + a^4$ by $x^2 - 2ax + a^2$.
20. $x^4 - 2ax^3 + 2a^3x - a^4$ by $x^2 - a^2$.
21. $a^3 + 3a^2 + 3a + 1$ by $a^2 + 2a + 1$.
22. $x^3 - 3x^2 + 3x + y^3 - 1$ by $x + y - 1$.
23. $x^8 - 3x^6 + 6x^4 - 7x^2 + 3$ by $x^4 - 2x^2 + 1$.

24. $18x^4 - 24x^3 + 38x^2 - 68x + 32$ by $6x - 4$.

25. $p^5 + p^4 + 4p^2 - 9p + 3$ by $p^3 + p^2 - 3p + 1$.

26. $20x^4 - 51x^3 - 12x^2 + 32x$ by $4x^2 - 7x - 8$.

27. 1 by $1 - x$, carrying the quotient to 6 terms.

28. $30x^4 - 130x^3 + 165x^2 - 147x + 36$ by $6x - 18$.

29. $21y^4 - 78y^3 - 17y^2 + 58y + 16$ by $7y^2 - 5y - 2$.

30. $x^5 + 7x^3y^2 - 5x^4y - x^2y^3 + 2y^5 - 4xy^4$ by $(x - y)^3$.

31. $-a^5 - 2a^4 + 2a^3 + 6a^2 + a - 1$ by $-a^2 + a + 1$.

32. $x^3 + y^3 + z^3 - 3xyz$ by $x^2 + y^2 + z^2 - xy - yz - zx$.

33. $x^6 - x^5 + 2x^4 + 4x^3 - 7x^2 + 4x - 1$ by $x^2 + x - 1$.

34. $-a^6 + 8a^5b - 14a^4b^2 + a^3b^3 + 6a^2b^4$ by $a^3 - 3a^2b + 2ab^2$.

35. $60x^5 - 85x^4 + 86x^3 - 69x^2 + 32x - 10$ by $3x^2 - 2x + 1$.

36. $26a^2 + 4a^3 - 3a^4 + a^5 - 92a + 55$ by $a^2 - 3a + 11$.

37. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.

38. $24m^4 - 14m^3 - 9m^2 - 84 + 43m$ by $7 - 3m + 4m^2$.

39. $x^4 + x^2y^2 + y^4$ by $y^2 - xy + x^2$. (Rearrange the divisor.)

40. $x^8 - 3x^7 - 5x^5 + 2x^4 + 5x^3 + 4x^2 + 2$ by $x^3 + 2x - 1$.

41. $3m^6 + 7m^5 - 12m^4 + 2m^3 - 3m^2 + 13m - 6$ by $m^2 + 3m - 2$.

42. $x^7 - x^6 - 2x^5 + 5x^4 - 5x^3 + 8x^2 + 6x - 12$ by $x^3 - 2x^2 + 3$.

43. $x^8 + 2x^7 + 3x^6 + 4(x^5 + 1) + 5x^4 + 6x^3 + 7x^2 + 8x$ by $(x + 1)^2$.

44. $x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1$ by $x^5 + x^4 + x^3 + x^2 + x + 1$.

45. Divide the product of $(x - 1)(x - 2)(x - 3)(x - 4)$ by $2(4 - 3x) + x^2$.

REVIEW EXERCISE XXVIII

1. Solve the equation $2 - (3 - \overline{4 - x}) = 3$.

2. Solve the equation $-2x + 4 = -12$.

3. Solve the equation $\frac{2}{3}x + 4 = \frac{1}{3}x + 4\frac{2}{3}$.

4. What are the advantages in detaching the coefficients when practicable?

5. Fifteen times a certain number, subtracted from 61, equals 1. Find the number.

6. What is the sign of the product of an odd number of negative numbers? Why?

7. If from twice a certain number we subtract 7 the result is 15. Find the number.

8. Three times a certain number, subtracted from 5, equals -10. Find the number.

9. Why do you avoid using such an arbitrary value in checking division as shall make the divisor zero?

10. If to three times a certain number we add 2, the result is five times the number. Find the number.

11. What is the value of

$$a\{a - b[a^2 - 2c(b^3 - \overline{a - b} \div c) + b] - c\}$$

when $a = 3$, $b = 1$, $c = 2$?

12. What is the Index Law of Division? Prove it for positive integers.

13. Solve the equation

$$x^2 + 2x - (x^2 - 2x) = \frac{10x + 20}{5}.$$

14. Distinguish between terms and factors; between coefficients and exponents.

CHAPTER V

FACTORS

I. DEFINITIONS AND TYPE FORMS

78. An algebraic expression is said to be **rational** with respect to any letter when, after simplifying, it contains no indicated root of that letter. In the contrary case it is said to be **irrational** with respect to that letter.

E.g., $4a + \sqrt{2}$ is rational with respect to a,
but $2 + 4\sqrt{a}$ is irrational with respect to a.

79. A rational algebraic expression is said to be **integral** with respect to a letter when this letter does not appear in any denominator. In the contrary case it is said to be **fractional.**

E.g., $\frac{1}{2} - \frac{a}{3}$ is an integral *algebraic* expression, with respect to a,

but $2 - \frac{3}{a}$ is a fractional expression, with respect to a.

So $x^2 - \frac{x}{a} + \frac{1}{a^2}$ is integral with respect to x, but not with respect to a,

but $x^2 - \sqrt{x}$ is not integral, because it is not rational.

80. In the chapter on Factoring, only rational integral algebraic expressions are considered, unless the contrary is stated, and the word "expression" is to be understood in this sense.

81. The factors of an algebraic expression are the algebraic expressions which multiplied together produce it.

In the expression $3x(x+1)(x^2+x+1)(x^3+2)$

3 is called a **numerical factor,**

x " " **monomial algebraic factor** of the first degree,

$x+1$ " " **binomial factor** of the first degree,

x^2+x+1 " " **quadratic trinomial factor,** the term "quadratic" being applied to integral algebraic expressions of the second degree in some letter or letters.

x^3+2 is called a **cubic binomial factor,** the term "cubic" being applied to integral algebraic expressions of the third degree in some letter or letters.

If an expression has no factors, it is said to be **prime.**

82. Factoring is the inverse of multiplication, and like all inverse processes it depends on a knowledge of the direct process and of certain type forms already known.

E.g., because we know that

$$(x+y)^2 = x^2 + 2xy + y^2,$$

therefore we know that the factors of

$$x^2 + 2xy + y^2 \text{ are } x+y \text{ and } x+y,$$

and those of m^2+2m+1 " $m+1$ " $m+1$.

In the same way, because we know that

$$(a+b)(a-b) = a^2 - b^2,$$

we know that the factors of a^2-b^2 are $a+b$ and $a-b$.

There are only a few type forms ordinarily used in factoring, and these will now be considered.

83. Type I. A monomial factor.

Type form, $\quad xy + xz.$

Since $x(y+z) = xy + xz$, it follows that expressions in the form of $xy + xz$ can be factored.

E.g., $4x^2 + 2x = 2x(2x+1)$. *Check.* $6 = 2 \cdot 3$.

A polynomial may often be treated as a monomial, as in the second step of the following:

$$\begin{aligned} y^2 - my + ny - mn &= y(y-m) + n(y-m) \\ &= (y+n)(y-m). \end{aligned}$$

Check. Let $y = 2$, $m = n = 1$. Then $3 = 3 \cdot 1$.

It must be remembered that *an expression is not factored unless it is written as a single product, not as the sum of several products.*

E.g., the preceding expression is not factored in the first step; only some of its *terms* are factored.

EXERCISE XXIX

Factor the following expressions:

1. $x^7 + x^6y + x^4$.
2. $a^3 + ab^2 - ac^2$.
3. $x^3 - x^4 - x^2 + x$.
4. $a^2 + 2ab + 3ac$.
5. $x^3y + xy^2z + x^2z$.
6. $3x^9 - 4ax^6 + x^5$.
7. $6a^2 + 6ab + 6b^2$.
8. $a^3b^2 + a^2b^3 + ab^4$.
9. $a^3b + a^2b^2 + a^2bc$.
10. $x^4y + x^3y^2 + x^2y^3$.
11. $ac + bc + ad + bd$.
12. $eg + fg + fh + eh$.
13. $9y^7 + 3by^5 + 6cy^4$.
14. $a^2xy + b^2xy + c^2xy$.
15. $m^3 + 3m^2n + 3mn^2$.
16. $aby - ay + y^2 - by$.
17. $abmn + cdmq + rm$.
18. $a^4m + a^2b^3m + a^2c^4m$.
19. $p^2q^2r^3 + pq^4r^3 + pq^2r^6$.
20. $w^2 - wy + wx - wxy$.
21. $a^2x^2 + a^2y + bx^2 + by$.
22. $a^2c^2 + a^2d - 2c^2 - 2d$.
23. $a^3b^3 + a^4b^2 + a^3b + a^2b^2$.
24. $abm^4n + bcm^2n^3 + cdm^3n$.

84. Type II. The square of a binomial.

Type form, $\quad \mathbf{x^2 \pm 2xy + y^2}.$

Since $(x \pm y)^2 = x^2 \pm 2xy + y^2$ (§ 68, 1, 2), it follows that expressions in the form of $x^2 \pm 2xy + y^2$ can be factored.

E.g., $\quad x^2 + 4x + 4 = (x+2)^2. \qquad$ *Check.* $\quad 9 = 3^2.$

$9x^2 - 6xy + y^2 = (3x - y)^2.$

EXERCISE XXX

Factor the following expressions:

1. $a^2 + 4a + 4.$
2. $a^2 + 14a + 49.$
3. $9p^2 + 6p + 1.$
4. $4x^2 + 4xy + y^2.$
5. $x^2 + 10x + 25.$
6. $9x^2 - 42x + 49.$
7. $25 + x^2 - 10x.$
8. $x^2 + 169 - 26x.$
9. $x^2 + 81 - 18x.$
10. $m^6 + 14m^3 + 49.$
11. $x^2 - 4xy + 4y^2.$
12. $a^2x^2 + y^2 - 2axy.$
13. $a^2x^2 + 2ax + 1.$
14. $x^2 + 16xy + 64y^2.$
15. $4m^2 - 12m + 9.$
16. $4x^2 + 4y(y - 2x).$
17. $x^4 - 22x^2 + 121.$
18. $4x^2 + 28xy + 49y^2.$
19. $121x^2 - 22x + 1.$
20. $9p^2 + 24pq + 16q^2.$
21. $9x^2 - 24xy + 16y^2.$
22. $81x^4 + 72x^2y^2 + 16y^4.$
23. $49z^2 + 81w^2 - 126zw.$
24. $(x+y)^2 + 2(x+y) + 1.$
25. $a^2b^2x^2y^2 - 4abxy + 4.$
26. $169a^2 + 169b^2 - 338ab.$
27. $a^2 + 4a + 4 + 2(a+2) + 1.$
28. $a^2 + 2ab + b^2 + 2(a+b)y + y^2.$
29. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2.$

85. Type III. The difference of two squares.

Type form, $\mathbf{x^2 - y^2}$.

Since $(x+y)(x-y) = x^2 - y^2$ (§ 68, 3), it follows that expressions in the form of $x^2 - y^2$ can be factored.

E.g., $x^2 - 16 = (x+4)(x-4)$. *Check.* $-15 = 5 \cdot -3$.

also
$$x^4 - 16 = (x^2+4)(x^2-4)$$
$$= (x^2+4)(x+2)(x-2).$$

Check. $-15 = 5 \cdot 3 \cdot -1$.

EXERCISE XXXI

Factor the following expressions:

1. $x^8 - y^2$.
2. $a^6 - b^2$.
3. $a^2 - 4b^2$.
4. $25a^4 - 1$.
5. $1 - 36x^4$.
6. $64x^4 - 9$.
7. $x^4 - 16b^4$.
8. $a^2b^2c^2 - 1$.
9. $16a^6 - b^2$.
10. $a^2b^4c^6 - d^2$.
11. $169a^2 - 1$.
12. $64a^2 - 81$.
13. $a^4b^2 - x^2y^4$.
14. $49x^{10} - 81$.
15. $121x^6y^6 - z^2$.
16. $a^2b^2c^2d^2 - 64$.
17. $36a^2b^2 - 9c^2$.
18. $100a^4 - 36b^2$.
19. $169x^2y^2 - 625$.
20. $144a^2 - 121b^2$.
21. $16x^4 - 81x^4y^4z^4$.
22. $49x^2y^4 - 16x^4y^2$.

86. There are special cases of the above type that deserve attention. One is that of the difference of the squares of polynomials, as in the case of

$$a^2 + 2ab + b^2 - x^2 - 2xy - y^2.$$

Here we have

$$a^2 + 2ab + b^2 - x^2 - 2xy - y^2 = (a+b)^2 - (x+y)^2$$
$$= (a+b+x+y)(a+b-x-y).$$

Check. $1 + 2 + 1 - 1 - 2 - 1 = 4 \cdot 0$.

Another special case is that of polynomials of the form

$$x^4 + x^2y^2 + y^4.$$

Here we see that

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2, \text{ the difference of two squares,} \\ &= (x^2+y^2+xy)(x^2+y^2-xy). \quad \textit{Check.} \quad 3=3\cdot 1. \end{aligned}$$

EXERCISE XXXII

Factor the following expressions:

1. $16x^4 + 4x^2y^2 + y^4$.
2. $81x^4 + 9x^2 + 1$.
3. $x^4 + 4x^2y^2 + 16y^4$.
4. $x^4 + 9x^2y^2 + 81y^4$.
5. $a^4 + 16a^2b^2 + 256b^4$.
6. $a^4 + 25a^2b^2 + 625b^4$.
7. $625x^4 + 25x^2y^2 + y^4$.
8. $16a^4b^4 + 4a^2b^2c^2 + c^4$.
9. $1296x^4 + y^4 + 36x^2y^2$.
10. $16p^4 + 81q^4 + 36p^2q^2$.
11. $16x^4 + 36x^2y^2 + 81y^4$.
12. $81x^4+225x^2y^2+625y^4$.
13. $x^2 - y^2 + 9 + 6x$.
14. $a^2 - 1 + b^2 + 2ab$.
15. $81a^4b^4c^4 + 9a^2b^2c^2d^2 + d^4$.
16. $16x^4 + 625y^4 + 100x^2y^2$.
17. $81m^4 + 256n^4 + 144m^2n^2$.
18. $625p^4 + 400p^2q^2 + 256q^4$.
19. $81x^4y^4z^4 + 225x^2y^2z^2 + 625$.
20. $16p^4q^4r^4 + 36p^2q^2r^2s^2 + 81s^4$.
21. $x^2 - a^2 + y^2 - b^2 + 2ab - 2xy$.
22. $x^2 + 2xy - z^2 + 2wz - w^2 + y^2$.
23. $a^2 - x^2 + 4b^2 - 4y^2 - 4ab - 4xy$.

87. Type IV. The cube of a binomial.

Type form, $\mathbf{x^3 \pm 3\,x^2y + 3\,xy^2 \pm y^3}$.

Since $(x \pm y)^3 = x^3 \pm 3\,x^2y + 3\,xy^2 \pm y^3$ (§ 68, 4, 5), it follows that expressions in the form of $x^3 \pm 3\,x^2y + 3\,xy^2 \pm y^3$ can be factored.

E.g., $8\,x^3 + 12\,x^2 + 6\,x + 1 = (2\,x)^3 + 3\,(2\,x)^2 + 3 \cdot 2\,x + 1$

$= (2\,x + 1)^3$. *Check.* $27 = 3^3$.

$$27\,x^6 - 54\,x^4y + 36\,x^2y^2 - 8\,y^3 = (3\,x^2)^3 - 3\,(3\,x^2)^2 \cdot 2\,y + 3 \cdot 3\,x^2(2\,y)^2 - (2\,y)^3$$

$= (3\,x^2 - 2\,y)^3$. *Check.* $1 = 1^3$.

EXERCISE XXXIII

Factor the following expressions:

1. $1 - 3\,x + 3\,x^2 - x^3$.
2. $a^3 - 3\,a^2 + 3\,a - 1$.
3. $x^{12} - 3\,x^8 + 3\,x^4 - 1$.
4. $x^3 - 9\,x^2 + 27\,x - 27$.
5. $27\,x^3 - 27\,x^2 + 9\,x - 1$.
6. $x^3 - 15\,x^2 + 75\,x - 125$.
7. $a^6 - 3\,a^4b^2 + 3\,a^2b^4 - b^6$.
8. $27\,a^9 - 27\,a^6 + 9\,a^3 - 1$.
9. $x^3 + 15\,x^2 + 75\,x + 125$.
10. $x^3 - 6\,x^2y + 12\,xy^2 - 8\,y^3$.
11. $8\,x^3 - 12\,x^2y + 6\,xy^2 - y^3$.
12. $54\,x^2 - 27\,x + 8\,x^4 - 36\,x^3$.
13. $x^3 - 9\,x^2y + 27\,xy^2 - 27\,y^3$.
14. $x^3 + 3\,ax^2y + 3\,a^2xy^2 + a^3y^3$.
15. $x^9y^6z^3 + 6\,x^6y^4z^2 + 12\,x^3y^2z + 8$.

88. Type V. Multiples of a binomial.

It often happens that expressions that do not come under any of the types already mentioned, contain a binomial factor. There is a simple method for determining this factor, suggested by a proposition called **The Remainder Theorem.** This theorem is as follows:

If an expression contains x, *the remainder arising from dividing it by* x − a *is the same as the expression with* a *put in place of* x.

For example, divide x^2+bx+c by $x-a$, and see if the remainder is the same as x^2+bx+c with a put in place of x.

$$\begin{array}{r|l} & x + a + b \\ x-a) & x^2 + bx + c \\ & \underline{x^2 - ax} \\ & (a+b)x + c \\ & \underline{(a+b)x - a^2 - ba} \\ & \qquad a^2 + ba + c \end{array}$$

The remainder is the same as x^2+bx+c, with a put in place of x.

It can be shown that this is true in all cases. Let D be the expression, q the quotient, and r the remainder. Then

1. $D = q(x-a)+r$, as in § 75.

(*I.e.*, the dividend equals the product of the quotient and the divisor, plus the remainder, and this is true whatever the value of x.)

2. But this step 1 being true whatever value we give to x, it is true if $x=a$.

3. This change does not affect r, because r has no x in it, for if it had, we would keep on dividing.

4. $\therefore\ D=q(a-a)+r=0+r=r$. That is, D becomes the same as r if we put a for x.

89. This proposition enables us to tell whether a binomial is a factor, without taking the time to divide, as will be seen in the following examples:

1. Is $x-1$ a factor of x^3-8x^2+9x-2?

Put 1 in place of x in the expression, and we have

$$1-8+9-2=0.$$

Hence, $x-1$ is a factor, and the other can be found by dividing.

2. Is $x-2$ a factor of x^3-4x^2+5x-2?

Put 2 in place of x, and we have

$$8-16+10-2=0.$$

Hence, $x-2$ is a factor, and x^2-2x+1 is the other factor, this being further factorable into $(x-1)^2$.

3. Is $x+1$ a factor of x^3-x^2-x+1?

Here $x+1$ is not of the type $x-a$, but it may be written $x-(-1)$. Then putting -1 in place of x we have

$$(-1)^3-(-1)^2-(-1)+1=-1-1+1+1=0.$$

Hence, $x+1$ is a factor. Dividing, x^2-2x+1, or $(x-1)^2$ is the other factor.

4. What are the factors of x^3+4x^2+4x+3?

Here we would naturally think, because the expression ends in 3, and contains no negative terms, that $x+1$ or $x+3$ would be a factor.

We may save ourselves the trouble of *unnecessary* division by using the Remainder Theorem. Putting -1 for x, as in Ex. 3, we have $-1+4-4+3$, which is *not* 0, hence $x+1$ is *not* a factor.

Putting -3 for x, we have

$$-27+36-12+3=0,$$

hence $x+3$ *is* a factor. Dividing, the other factor is

$$x^2+x+1.$$

EXERCISE XXXIV

Factor the following expressions:

1. p^3+p^2-2.
2. x^3-3x-2.
3. x^3-2x-4.
4. x^3-2x-1.
5. p^3-2p-4.
6. $x^3-2xy^2+y^3$.
7. $x^3-19x-30$.
8. $m^3-2mn^2+n^3$.
9. a^3-a^2-a-2.
10. x^3+3x^2+x-2.
11. x^3-5x^2+x-5.
12. x^3+x^2-7x-3.
13. p^3-p^2-4p-6.
14. $a^3-a-2+2a^2$.
15. x^3-2x^2-2x-3.
16. $a^3-a^2-15a+12$.
17. $a^3-6a^2+11a-6$.
18. $x^3+9x^2+20x+12$.

90. Type VI. Binomials.

Type form, $\mathbf{x^n \pm y^n}$.

Since $$(x+y)(x-y)=x^2-y^2,$$
it is evident that binomials of the form x^2-y^2 can be factored.

Since $$(x+y)(x^2-xy+y^2)=x^3+y^3,$$
and $$(x-y)(x^2+xy+y^2)=x^3-y^3,$$
it is evident that binomials of the form $x^3\pm y^3$ can be factored.

ILLUSTRATIVE PROBLEMS

1. Factor x^6+1.

$$\begin{aligned} x^6+1&=(x^2)^3+1\\ &=(x^2+1)[(x^2)^2-x^2+1]\\ &=(x^2+1)(x^4-x^2+1).\end{aligned}$$

2. Factor $(x+y)^3-z^3$.

$$\begin{aligned}(x+y)^3-z^3&=[(x+y)-z][(x+y)^2+(x+y)z+z^2]\\ &=(x+y-z)(x^2+2xy+y^2+xz+yz+z^2).\end{aligned}$$

3. Factor $8x^3-1$.

$$8x^3-1=(2x)^3-1=(2x-1)(4x^2+2x+1).$$

EXERCISE XXXV

Factor the following expressions:

1. $64x^3 - 1.$
2. $27x^6 - y^6.$
3. $125x^6 - 1.$
4. $8x^3 - 27y^3.$
5. $27x^3 + 125.$
6. $121x^2 - 25.$
7. $36x^2 - 25y^2.$
8. $81x^4 - 49y^2.$
9. $8(x+y)^3 + 1.$
10. $27(x-y)^3 - 1.$
11. $144x^4 - 121y^2.$
12. $216(x+y)^3 + z^3.$
13. $343x^3 - 125y^3.$
14. $729(x+1)^3 + 1.$
15. $(a+b+c)^2 - 1.$
16. $a^2 + 2ab + b^2 - c^2.$
17. $(x+y+z)^2 - 64.$
18. $(x^2+y)^2 - (x+y^2)^2.$
19. $100(a+b^2+c^3)^2 - b^4.$
20. $64(a^2+b)^2 - 25(a+b^2)^2.$

91. From the Remainder Theorem we see that if n is a positive integer,

(1) $x^n + y^n$ *is divisible by* $x + y$ *when* n *is odd.*

For, putting $-y$ for x, $x^n + y^n$ becomes $(-y)^n + y^n$, which equals 0 when n is odd, and not otherwise.

(2) $x^n + y^n$ *is never divisible by* $x - y$.

For, putting y for x, $x^n + y^n$ becomes $y^n + y^n$, which is not 0.

(3) $x^n - y^n$ *is divisible by* $x + y$ *when* n *is even.*

For, putting $-y$ for x, $x^n - y^n$ becomes $(-y)^n - y^n$, which equals 0 when n is even, and not otherwise.

(4) $x^n - y^n$ *is always divisible by* $x - y$.

For, putting y for x, $x^n - y^n$ becomes 0.

92. This may be summarized as follows:

$x^n + y^n$ contains the factor $x + y$ when n is *odd*,

" " " " $x - y$ *never*.

$x^n - y^n$ " " " $x + y$ when n is *even*.

" " " " $x - y$ *always*.

Hence, it follows that expressions in the form of $x^n \pm y^n$ can often be factored.

93. This type occurs so often that the forms of the quotients should be memorized:

1. $\dfrac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots$, the signs alternating.

2. $\dfrac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots$, the signs alternating.

3. $\dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \cdots$, the signs being all $+$.

We are thus able to write out the quotient of $(x^{15} + y^{15}) \div (x + y)$ at sight, and so for other similar cases.

The integral parts of the quotients in 1 and 2 are the same, but the remainders are different. *E.g.*, if n is odd there is no remainder in 1, but in 2 there is a remainder $-2y^n$.

From these quotients we can readily state the factors of certain binomials. For example, as in § 90,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2), \qquad \text{from step 3.}$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2), \qquad \text{" " 1.}$$

94. When the exponent n exceeds 3, it is better to separate into two factors as nearly of the same degree as possible, and then to factor each separately.

E.g.,
$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

This is better than to take out the binomial $x + y$ or $x - y$ first, which would give

$$x^8 - y^8 = (x + y)(x^7 - x^6y + x^5y^2 - x^4y^3 + x^3y^4 - x^2y^5 + xy^6 - y^7),$$

or
$$(x - y)(x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7),$$

in which cases it would be difficult to discover the factors of the two expressions of the seventh degree.

So
$$x^{2n} - y^{2n} = (x^n + y^n)(x^n - y^n).$$

95. Binomials of the form $x^n \pm y^n$ which have not the factor $x \pm y$ may contain $x^m \pm y^m$.

E.g., $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

EXERCISE XXXVI

Factor the following expressions:

1. $x^3 + 1$.	2. $x^3 - 1$.
3. $x^4 - 1$.	4. $x^7 + 1$.
5. $x^5 + 1$.	6. $1 - x^{16}$.
7. $x^5 + y^5$.	8. $x^6 - y^6$.
9. $x^4 - 16$.	10. $x^3 + 8y^3$.
11. $32x^5 + 1$.	12. $16x^4 - 1$.
13. $32x^5 + y^5$.	14. $64x^6 - 1$.
15. $32x^5 - 1$.	16. $x^{12} + 4096$.
17. $a^2b^4 - b^2a^4$.	18. $729x^6 + y^6$.

19. $216a^6 - b^3$.
20. $128a^7 + b^7$.
21. $125a^3 + 27$.
22. $(x+y)^3 + 1$.
23. $27a^3 + 64b^3$.
24. $27a^3b^3c^3 + 8$.
25. $(x+y)^3 - z^3$.
26. $x^6y^3 + 64w^9z^6$.
27. $243x^5y^5z^5 - 1$.
28. $128x^7y^7z^7 + 1$.
29. $64x^3 - 729y^6$.
30. $32a^5 + 243b^5$.
31. $a^6b^9c^{12} - x^3y^6z^9$.
32. $125x^7 - 27xy^3$.
33. $1331a^3b^6c^9 - 1$.
34. $256x^8y^{16} - 625$.
35. $a^3 + a + b^3 + b$.
36. $64a^4b^4 - 81c^4d^4$.
37. $256m^8n^8 - p^{16}q^{16}$.
38. $64x^3y^3z^3 + 125w^3$.
39. $343(x-y)^3 - 64$.
40. $m^2 - n^2 + 2n - 1$.
41. $8x^3 + 125(x+y)^3$.
42. $(a-b)^3 - (a+b)^3$.
43. $8a^3b^3c^3d^3 + 27e^6f^6g^6$.
44. $27x^3y^3z^3 + 64u^6v^6w^6$.
45. $(x^3+1)^2 - (x+1)^2$.
46. $(a^2+b)^3 + (a+b^2)^3$.
47. $(x^2+1)^3 + (x^2-1)^3$.
48. $(a^3+1)^3 - (b^3+1)^3$.

96. Type VII. The quadratic trinomial.

Type form, $\mathbf{x^2 + ax + b}$.

We have already met (§ 84) one quadratic trinomial, $x^2 + 2xy + y^2 = (x+y)^2$. The above type is more general, and the type $ax^2 + bx + c$ is more general still, and will be considered on page 84.

Let $x^2 + ax + b = (x+m)(x+n)$, in which m and n are to be determined. Then

$$x^2 + ax + b = x^2 + (m+n)x + mn.$$

It therefore appears that *if two numbers,* m *and* n, *can be found such that their sum,* m + n, *is* a, *and their product,* mn, *is* b, *the expression can be factored.*

E.g., consider $x^2 + 10x + 21$.

Here $\qquad 10 = 3 + 7$,

and $\qquad 21 = 3 \cdot 7$,

$\therefore\ x^2 + 10x + 21 = (x + 3)(x + 7)$. *Check.* $32 = 4 \cdot 8$.

Consider also $x^2 - 3x - 40$.

Here $\qquad -3 = 5 - 8$,

and $\qquad -40 = 5 \cdot -8$,

$\therefore\ x^2 - 3x - 40 = (x + 5)(x - 8)$.

Check. $-42 = 6 \cdot -7$.

EXERCISE XXXVII

Factor the following expressions:

1. $x^2 - x - 2$.
2. $x^2 + 3x + 2$.
3. $x^4 + x^2 - 12$.
4. $x^2 - 5x + 6$.
5. $p^2 - p - 600$.
6. $x^2 - 4x - 21$.
7. $x^2 + 8x - 33$.
8. $x^2 - 5x - 36$.
9. $x^2 + 4x - 77$.
10. $x^6 - 4x^3 - 45$.
11. $x^2 + 14x + 45$.
12. $x^2 - 13x + 36$.
13. $x^2 - 18x + 77$.
14. $x^2 - 16x + 39$.
15. $x^2 + 10x - 39$.
16. $x^2 - 12x - 85$.
17. $x^2 + 12x - 85$.
18. $a^2 + 17a + 66$.
19. $x^2 - 4x - 165$.
20. $a^2 - 3a - 130$.
21. $a^2 - 11a - 60$.
22. $x^2 - 12x + 35$.
23. $x^2 + 11x - 26$.
24. $x^4 + 16x^2 + 55$.
25. $x^2 + 41x + 420$.
26. $x^4y^2 + 4x^2y + 3$.
27. $a^2 - 24a + 135$.
28. $a^2 - 16a - 225$.
29. $a^4x^4 + 5a^2x^2 + 6$.
30. $x^4 - 15x^2 - 100$.
31. $x^2 + 7xy + 10y^2$.
32. $a^6x^2 - 5a^3x - 14$.
33. $m^2 - 38m + 165$.
34. $x^2 + 11xy - 26y^2$.
35. $m^2x^2 - 7mx - 18$.
36. $m^2x^4 + 12mx^2 + 35$.

97. Type form, $\mathbf{ax^2 + bx + c}$.

There are several methods of factoring this type. Two of the best are here given, it being recommended that the teacher and class adopt one of them and follow that.

Consider the trinomial $6x^2 + 17x + 12$. This may be written

$$\tfrac{1}{6}[36x^2 + 17 \cdot 6x + 72]$$

or

$$\tfrac{1}{6}[(6x)^2 + 17 \cdot (6x) + 72].$$

Now $z^2 + 17z + 72 = (z+9)(z+8)$.

$$\begin{aligned} \therefore\ \tfrac{1}{6}[(6x)^2 + 17 \cdot (6x) + 72] &= \tfrac{1}{6}(6x+9)(6x+8) \\ &= \tfrac{1}{6} \cdot 3(2x+3) \cdot 2 \cdot (3x+4) \\ &= (2x+3)(3x+4). \end{aligned}$$

And, in general, we may *arrange the trinomial in the form*

$$z^2 + pz + q.$$

98. Or we may proceed as follows: Let

$$ax^2 + bx + c = (mx+n)(px+q),$$

in which, m, n, p, and q are to be determined. Then

$$ax^2 + bx + c = mpx^2 + (mq + pn)x + qn.$$

It therefore appears that *the coefficient of* x, $mq + pn$, *is the sum of two numbers whose product*, $mqpn$, *is the product of the coefficient of* x^2, mp, *and the last term*, qn. Hence, if these numbers can be detected, the expression can be factored.

E.g., consider $6x^2 + 17x + 12$, as in § 97.

Here $17 = 9 + 8$,

and $6 \cdot 12 = 72 = 9 \cdot 8$.

$$\begin{aligned} \therefore\ 6x^2 + 17x + 12 &= 6x^2 + 9x + 8x + 12 \\ &= 3x(2x+3) + 4(2x+3) \\ &= (3x+4)(2x+3). \end{aligned}$$

Check. $35 = 7 \cdot 5$.

Consider also $6x^2 + 7x - 3$.

Here $\quad 7 = 9 - 2,$

and $\quad 6 \cdot -3 = -18 = 9 \cdot -2.$

$$\begin{aligned}\therefore\ 6x^2 + 7x - 3 &= 6x^2 + 9x - 2x - 3\\ &= 3x(2x+3) - (2x+3)\\ &= (3x-1)(2x+3).\end{aligned}$$

Check. $10 = 2 \cdot 5$.

EXERCISE XXXVIII

Factor the following expressions:

1. $6x^2 + x - 12$.

2. $12p^2 - p - 1$.

3. $6x^2 + x - 15$.

4. $4x^2 - 4x - 3$.

5. $3a^2 + 8a + 4$.

6. $600a^2 - a - 1$.

7. $6a^2 + 7a - 49$.

8. $84x^2 - 5x - 1$.

9. $12p^2 - 7p + 1$.

10. $7x^2 - 50x + 7$.

11. $9x^2 - 17x - 2$.

12. $2m^2 - 3m - 20$.

13. $3m^2 - 2m - 21$.

14. $15x^2 - 23x + 4$.

15. $8a^2 + 22a + 12$.

16. $12x^2 + 7x - 12$.

17. $2p^2 - 5pq + 2q^2$.

18. $7x^2 + 4xy - 3y^2$.

19. $16x^2 - 62x + 27$.

20. $6p^2 + 25(p+1)$.

21. $a^{12} - 7a^6b^6 - 8b^{12}$.

22. $10x^2 - 47x + 42$.

23. $6p^2 + 25pq + 14q^2$.

24. $7x^2 - 12xy + 5y^2$.

25. $12a^2 - 25ab + 12b^2$.

26. $6a^2 + 13ab + 6b^2$.

27. $16a^2 + 43ab + 27b^2$.

28. $30x^2 - 41xz - 15z^2$.

29. $x^4y^2 + 17x^2yz + 72z^2$.

30. $25x^2y^2 - 5xyz - 6z^2$.

31. $35m^2 - 16mn - 3n^2$.

32. $40a^2 + 61ab - 84b^2$.

33. $30m^2 - 61mn + 30n^2$.

34. $35x^2y^2z^2 + 11xyz - 6$.

35. $x^6y^4z^2 + 15x^3y^2zw + 36w^2$.

36. $16x^2y^2z^2 + 39xyz - 27$.

99. Forms of the factors. Although an algebraic expression admits of only one distinct set of prime factors, the forms of these factors may often appear to differ.

E.g., since $(x-2y)(2x-y)=2x^2-5xy+2y^2$,

and $(2y-x)(y-2x)=2x^2-5xy+2y^2$,

it might seem that $2x^2-5xy+2y^2$ has two distinct pairs of factors.

This arises from the fact that the second pair is the same as the first, except that the signs are changed, each factor having been multiplied by -1. But this merely multiplies the whole expression by $-1\cdot-1$, that is, by $+1$.

Hence, *the signs of any even number of factors may be changed without changing the product.*

E.g., $x^2-5x+6=(x-2)(x-3)$, or $(2-x)(3-x)$.

Check. $2=-1\cdot-2$, or $1\cdot2$.

$$\begin{aligned}x^4-1&=(x^2+1)(x+1)(x-1)\\&=(x^2+1)(-x-1)(1-x)\\&=(-x^2-1)(x+1)(1-x).\end{aligned}$$

Check. Let $x=2$. Then

$$16-1=5\cdot3\cdot1=5\cdot-3\cdot-1=-5\cdot3\cdot-1.$$

EXERCISE XXXIX

Factor the following, giving the various forms of the results:

1. $1-a^3$.
2. x^3-1.
3. $16-x^4$.
4. a^6-b^6.
5. $16x^4-81y^4$.
6. x^6-64y^6.
7. $121+x^2-22x$.
8. $16m^4-81n^4$.
9. $x^{10}-26x^5+168$.
10. $a^2-c^2+b^2+2ab$.

MISCELLANEOUS EXERCISE XL

100. *General directions.*

1. First remove all monomial factors.

2. Then see if the expression can be brought under some of the simple types given on pp. 71–84. This can probably always be done in cases of binomials and quadratic trinomials, and often in other cases.

3. If unsuccessful in this, the Remainder Theorem may be tried, especially with polynomials of the form

$$x^n + ax^{n-1}y + bx^{n-2}y^2 + \cdots.$$

4. Always be sure that the factors are prime.

Factor the following expressions:

1. $x^6 + y^6$.	2. $1 + x^2 + x^4$.
3. $y^8 - x^4y^4$.	4. $x^8 + y^8 + x^4y^4$.
5. $x^4 + x^2 + \frac{1}{4}$.	6. $x^4 - 2x^2y^2 + y^4$.
7. $a^8b^6c^4 - a^4b^6c^8$.	8. $x^2(x^2 + y^2) + y^4$.
9. $a^6 - a^3 - 110$.	10. $x^4 + x^2y + 2x^3y$.
11. $x^4 - 11x^2 + 1$.	12. $2x^2 + 11x + 12$.
13. $6x^2 - 23x + 20$.	14. $y^2 - z^2 + 2z - 1$.
15. $x^3y^3 + 2x^2y^2 + xy$.	16. $(x + y)^7 - x^7 - y^7$.
17. $a^4 - 15a^2b^2 + 9b^4$.	18. $ax^2 + (a + b)x + b$.
19. $x^4 - 8x^2y^2 + 16y^4$.	20. $12x^2y^2 - 17xy + 6$.
21. $ab + y^2 - ay - by$.	22. $(x + 1)^2 - 5x - 29$.
23. $(a + b)^3 + (a - b)^3$.	24. $16x^4 - 28x^2y^2 + y^4$.
25. $y^3 + 3y^2 + 6y + 18$.	26. $7x^3 + 96x^2 - 103x$.
27. $21a^2 + 26ab - 15b^2$.	28. $x^4 - (a^2 + b^2)x^2 + a^2b^2$.

29. $m^6n^6+1.$

30. $x^4+6x^2-7.$

31. $9x^2-16y^2.$

32. $6x^2+7x-55.$

33. $x^4+x^2-20.$

34. $a^8-2a^4b^4+b^8.$

35. $x^{4m}+x^{2m}+1.$

36. $9a^{2m}-5-4a^m.$

37. $6x^4+x^2-2.$

38. $m^3+3m^2+2m.$

39. $x^4-4x^2-21.$

40. $22x^2+53x-5.$

41. $a^5+a-2a^3.$

42. $x^2-5xy-14y^2.$

43. $x^4+a^8+x^2a^4.$

44. $x^{2n}-11x^n+28.$

45. $10a^2-360b^4.$

46. $a^2(a^2-24)+63.$

47. $x^2+16x+63.$

48. $a^2-ac-bc-b^2.$

49. $x^2-14x+49.$

50. $16x^4+y^4+4x^2y^2.$

51. $x^4-21x^2+38.$

52. $x^2+12xy+36y^2.$

53. $a^3(a^3-1)-56.$

54. $p^2-20pq+91q^2.$

55. $8-(x+y+z)^3.$

56. $44x^2+137x+44.$

57. $6+15a^2-19a.$

58. $28x^2y^2+17xy-3.$

59. $x^2-3xy-28y^2.$

60. $m^2-t^2-w^2+2tw.$

61. $xy+yz+y^2+xz.$

62. $(a^2+1)^3-(b^2+1)^3.$

63. $21x^2-43x+20.$

64. $(x+y)^4+4(w+z)^4.$

65. $9x^2-82xy+9y^2.$

66. $3x^2+5xyz-12y^2z^2.$

67. $10x^2y^2+33xy-7.$

68. $a^3b-ab^3+a^2b+ab^2.$

69. $143b^4+a^2+24ab^2.$

70. $4a^2+44ab^2+121b^4.$

71. $3xy(x+y)+x^3+y^3.$

72. $5ab-bc+cd-5ad.$

73. $a^2(a+1)-b^2(b+1).$

74. $x^3+y^3-4x^2y-4xy^2.$

75. $a^4-16c^2+b^2+2a^2b.$

76. $6x^2-43xyz-68y^2z^2.$

77. $4x^2y^2-(x^2+y^2-z^2)^2.$

78. $w^2x^2y^2z^2-wxyz-132.$

79. $28b^2c^2+9a^4-33a^2bc.$

80. $4m^4+28m^2n^3+49n^6.$

81. $16x^4+8x^2+1-25y^6.$

82. $(a-4)^2-4(a-4)+4.$

83. $xw+2wy+2xz+4yz.$

84. $3m^4n+3m^2n^3+3m^3n^2.$

85. $x^3(x-2y)-y^3(y-2x).$

86. $(x-5)^2-8(x-5)+12.$

87. $4a^4b^2+9x^2y^4-12a^2bxy^2.$

88. $mx+3nx+9ny+3my.$

89. $-x^4-15x^3+23x^2-7x.$

90. $(a+b)^2+49-14(a+b).$

91. $9a^4+49b^4c^2d^2-42a^2b^2cd.$

92. $1-(a-b)-110(a-b)^2.$

93. $10+16(a+b)+6(a+b)^2.$

94. $5am-7bn+5bm-7an.$

95. $49a^2b^2c^2d^2-119abcd+66.$

96. $(m+n)^2+10(m+n)+24.$

97. $2x^2-x^2y+(y-2)(xy-x)^2.$

98. $121x^2+121y^2-9-242xy.$

99. $x^2+y^2+9(x+y)+2xy+18.$

100. $4x^2+1-y^2-2yz-z^2+4x.$

101. $a^2-4a+b^2-4b-77+2ab.$

102. $a^2+4a+b^2-4b-77-2ab.$

103. $x^2+y^2-(w^2+z^2)+2(xy+wz).$

104. $p^2+12p+q^2+12q+2pq+35.$

II. APPLICATION OF FACTORING TO THE SOLUTION OF EQUATIONS

101. A simple, interesting, and valuable application of factoring is found in the solution of equations of higher degree than the first.

To solve an equation is to find the value of the unknown quantity which shall make the first member equal to the second. Such a value is said to *satisfy* the equation (§ 17). For example, consider the equation

$$x^2 + x = 6.$$

We have $\qquad x^2 + x - 6 = 0 \qquad$ by subtracting 6,

whence $\qquad (x+3)(x-2) = 0.$

This equation is evidently satisfied if *either* factor of the first member is 0, because the product would then be 0.

If $\quad x + 3 = 0$, then $x = -3$, because $-3 + 3 = 0$;

and if $\quad x - 2 = 0$, " $x = 2$, " $2 - 2 = 0$.

Again, consider the equation

$$x^4 - 6x^3 + 11x^2 - 6x = 0.$$

Factoring, we have

$$x(x-1)(x-2)(x-3) = 0.$$

This equation is evidently satisfied if *any* factor of the first member equals 0, because the product would then be 0.

Hence, x may equal 0, as one value;

or if $\quad x - 1 = 0$, then $x = 1$, because $1 - 1 = 0$;

and if $\quad x - 2 = 0$, " $x = 2$, " $2 - 2 = 0$;

and if $\quad x - 3 = 0$, " $x = 3$, " $3 - 3 = 0$.

EXERCISE XLI

Solve the following equations:

1. $x^2-1=0.$
2. $x^2-x=12.$
3. $x^2-x=6.$
4. $x^2+5x=0.$
5. $x^2-4x=0.$
6. $x^2-2x=8.$
7. $x^2-x=30.$
8. $x^2+4x=5.$
9. $2x^2+2=5x.$
10. $x^2+2x=15.$
11. $x^2-4x=21.$
12. $x^2-6x=27.$
13. $x^2+3x=10.$
14. $x^2+287=48x.$
15. $x^2=2x+143.$
16. $x^2+6x-27=0.$
17. $x^2-5x+6=0.$
18. $x^2+2x-99=0.$
19. $x^3+4x^2+x=6.$
20. $x^2-7x+12=0.$
21. $x^2-8x+7=0.$
22. $x^2-12x+35=0.$
23. $x^2+5x-50=0.$
24. $x^2-18x+65=0.$
25. $x^2-10x+9=0.$
26. $x^2-15x+26=0.$
27. $x^2+5x-36=0.$
28. $x^2+11x-26=0.$
29. $x^2-2x-99=0.$
30. $x^2-14x+49=0.$
31. $x^2+18x+81=0.$
32. $x^2+16x+64=0.$
33. $x^2-15x+44=0.$
34. $6x^2-13x+6=0.$
35. $x^4-13x^2+36=0.$
36. $x^2-10x+21=0.$
37. $x^2-11x+30=0.$
38. $x^6-14x^4+49x^2=36.$
39. $x^2-22x+121=0.$
40. $2x^3-67x^2+371x=0.$
41. $2x^3-7x^2+5x=0.$
42. $x^4-15x^2+10x+24=0$

REVIEW EXERCISE XLII

1. Solve the equation $x(x-3)=0$.

2. Factor $64x^6y^6-48x^4y^4+12x^2y^2-1$.

3. Factor $1.331x^3-7.26x^2+13.2x-8$.

4. Factor $x^3-3abcx^2+3a^2b^2c^2x-a^3b^3c^3$.

5. Divide $np+nq+mq+mp$ by $m+n$.

6. Factor $(a+b)^3+3(a+b)^2+3(a+b)+1$.

7. Divide $39a-91c+26b$ by $2b+3a-7c$.

8. Solve the equation $x(x-5)(x-6)(x+3)=0$.

9. If $x=0$, $y=427$, $z=3281$, what is the value of $3x^4y^2z^3$.

10. From $10x^4-3x^3+4x^2-x+2$ subtract $2+x-4x^2+3x^3-10x^4$.

11. Perform the multiplication indicated in abc when $a=x+y$, $b=x-y$, $c=x^2+y^2$.

12. What does az become when $b+c$ is substituted for a and $x+y$ is substituted for z? Expand.

13. In the expression $5(a-b)^2(a+b)^3$, what coefficient and exponents are expressed? What are the factors?

14. Add $x^3-3x^2y+3xy^2-y^3$, $y^3-3y^2z+3yz^2-z^3$, $z^3-3z^2x+3zx^2-x^3$, $3[yz(y+z)+zx(z+x)+xy(x+y)]$.

15. Express algebraically the sum of 17 times one number and 15 times the product of that number by the cube of another number.

16. Multiply the sum of $x^2-4xy+3y^2$ and $4xy-2y^2$ by the difference between the minuend $5x^2-8xy+7y^2$ and the subtrahend $8y^2+4x^2-8xy$, and factor the result.

CHAPTER VI

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

I. HIGHEST COMMON FACTOR

102. The algebraic factor of highest degree common to two or more algebraic expressions is called their **highest common factor.**

E.g., a^2	is the highest common factor of	a^3cd	and $2a^2be^3$,
xy^2z	" " "	x^3y^2z	" xy^2z^3,
$a-b$	" " "	$(a-b)^2$	" a^2-b^2.

The word "expression" is here taken with the same limited meaning as in § 80.

103. Algebraic expressions sometimes *appear* to have several highest common factors. Thus (§ 99),

$$a^3-b^3=(a-b)(a^2+ab+b^2) \text{ or } -(b-a)(a^2+ab+b^2),$$

$$\text{and } b^2-a^2=-(a-b)(a+b) \qquad \text{or} \qquad (b-a)(b+a).$$

Here *either* $a-b$ or $b-a$ is a common factor, and they are of the same degree (same "height"), and there is no other common factor, hence each is the highest common factor. They are, however, the same in absolute value. In these cases either answer is accepted as correct.

104. The arithmetic greatest common divisor must not be confounded with the algebraic highest common factor, although these are often called by the same name. The highest common factor has reference only to the degree of the expression.

For example, the highest common factor of a^2 and ab is evidently a. But if $a=2$ and $b=4$, the greatest common divisor of 2^2 and $2\cdot 4$ is not $a=2$; it is 4.

105. Factoring method. The highest common factor of expressions which are easily factored is usually found by simple inspection.

E.g., to find the highest common factor of x^2-3x+2, x^3-x^2-2x, and x^2+x-6, we have:

1. $x^2-3x+2=(x-2)(x-1)$.
2. $x^3-x^2-2x=x(x-2)(x+1)$.
3. $x^2+x-6=(x-2)(x+3)$.
4. $\therefore$ the highest common factor is $x-2$, or $2-x$.

EXERCISE XLIII

Find the highest common factor of each of the following sets of expressions:

1. $a^2-b^2,\ a^3-b^3$.
2. $5a^2b^3c^4d^5,\ a^5b^4c^3d^2$.
3. $x^2+x-2,\ x^2+8x-9$.
4. $15mnx^3,\ 17mx^2yz,\ abcx^{10}z$.
5. $10x^2yz,\ 15ax^2yz^3,\ 20amxz^{10}$.
6. $x^2-7x+10,\ x^2-12x+35$.
7. $x^2+13x+40,\ x^2-4x-45$.
8. $x^3-y^3,\ y^2-x^2,\ 2x^2-xy-y^2$.

9. $x^2-49,\ x^3+9x^2+17x+21.$

10. $x^2+13x-30,\ x^2+11x-60.$

11. $x^2+14x+33,\ x^2+12x+27.$

12. $x^2-24x+143,\ x^2+4x-165.$

13. $x^2-y^2,\ y^3-x^3,\ x^2-8xy+7y^2.$

14. $x^3-15x^2+50x,\ x^2-16x+55.$

15. $6x^2+13x+6,\ 8x^2+22x+15.$

16. $x^2-4,\ x^2-x-6,\ 2-5x-3x^2.$

17. $2x^2-xy-y^2,\ 4x^2+10xy+4y^2.$

18. $4a^2(a^3-b^3),\ ab^2(3a^2-5ab+2b^2).$

19. $6a^2+19ab-7b^2,\ 2a^2+ab-21b^2.$

20. $x^4+x^2y^2+y^4,\ x^3+2x^2y+2xy^2+y^3.$

106. If the factors of one of several algebraic expressions are known, but those of the others not, it is easy to ascertain, by division or by the Remainder Theorem, if the known factors of the one are factors of the other.

E.g., to find the highest common factor of $1-x^2$ and $113x^7-4x^3+2x-111$.

Here $1-x^2=(1-x)(1+x)$, or $-(x-1)(x+1)$.

But $x-1$ is a factor of $113x^7-4x^3+2x-111$, by the Remainder Theorem (§ 88), while $x+1$ is not. $\therefore x-1$ is the highest common factor.

EXERCISE XLIV

Find the highest common factor of each of the following sets of expressions:

1. $x^5-y^5,\ x^2-y^2.$

2. $x^2-4,\ x^5-4x^2-16.$

3. $x^2-4,\ x^7+7x^2+100.$

4. $x^3+1,\ x^3+ax^2+ax+1.$

5. $x^2-3x+2,\ x^2-9x+14.$

6. $x^2-9x+14,\ 2x^3-5x^2-441.$

7. $x^3+x^2+x-3,\ x^3+3x^2+5x+3.$

8. $x^2-4,\ 5x^4+2x^3-23x^2-8x+12.$

9. $2x^2-5xy+3y^2,\ 6x^3-23x^2y+25xy^2-6y^3.$

10. $a^3-b^3,\ b^2-a^2,\ 117a^3-117a^2b-231ab+231b^2.$

11. $x^3-1,\ x^2-1,\ 293x^5-200x^4+7x^3-50x^2-25x-25.$

12. $1-x^4,\ x^5-1,\ x^3-1+3x-3x^2,\ 247x^2-240x-7.$

13. $x^5-32,\ 16-x^4,\ x^2-9x+14,\ x^4-4x^2+6x-12.$

14. $x^3+1,\ x^2+2x+1,\ x^5+1,\ 324x^5+247x^4+100x^3$
$+204x^2-27.$

107. Euclidean method. In case the highest common factor is not readily found by inspection of factors, a longer method, analogous to one suggested by Euclid (B.C. 300) for finding the greatest common divisor of two numbers, may be employed.

108. This method depends upon two theorems:

1. *A factor of an algebraic expression is a factor of any multiple of that expression.*

This is easily seen to be true for numbers. For example, 5 is a factor of 35; and since multiplying 35 by any integral number does not take out this 5, therefore 5 is a factor of any multiple of 35.

Similar reasoning holds in algebra; for if a factor is contained in an expression n times, it will be contained in m times this expression mn times.

2. *A factor of each of two algebraic expressions is a factor of the sum and of the difference of any multiples of those expressions.*

For if it is contained in the first m times, it will be contained in p times the first pm times; and if in the second n times, then in q times the second qn times; hence in their sum $pm + qn$ times, and in their difference $pm - qn$ times.

109. The Euclidean method will best be understood by considering an example.

Required the highest common factor of

$$x^4 - x^3 + 2x^2 - x + 1 \text{ and } x^4 + x^3 + 2x^2 + x + 1.$$

$$\begin{array}{l}
x^4-x^3+2x^2-x+1\,|\,x^4+x^3+2x^2+x+1\,|\,\underline{1} \\
\qquad\qquad\qquad\qquad\;\; \underline{x^4-x^3+2x^2-x+1} \\
\qquad\qquad\qquad\qquad\;\; 2x\,|\,\underline{2x^3 \qquad\quad +2x} \\
\qquad\qquad\qquad\qquad\qquad\;\; x^2 \qquad\quad +1\,|\,x^4-x^3+2x^2-x+1\,|\,\underline{x^2-x+1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^4 \qquad +x^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -x^3+x^2-x+1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-x^3 \qquad\; -x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x^2 \quad +1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2 \quad +1}
\end{array}$$

Explanation. 1. The h. c. f. of the two expressions is also a factor of $2x^3 + 2x$, by th. 2 (§ 108).

2. It cannot contain $2x$, because that is not common to the two expressions.

3. $\therefore$ $2x$ may be rejected, and the h. c. f. must be a factor of $x^2 + 1$.

4. $x^2 + 1$ is a factor of $x^4 - x^3 + 2x^2 - x + 1$, by trial.

5. " " " $2x^3 + 2x$.

6. $\therefore$ " " " $x^4 + x^3 + 2x^2 + x + 1$. (Why?)

7. $\therefore$ " is the h. c. f. (Why?)

110. In order to avoid numerical fractions in the divisions, it is frequently necessary to introduce numerical factors. These evidently do not affect the *degree* of the highest common factor.

E.g., to find the highest common factor of $4x^3 - 12x^2 + 11x - 3$ and $6x^3 - 13x^2 + 9x - 2$.

$$
\begin{array}{r|l|l}
 & 6x^3 - 13x^2 + 9x - 2 & \\
 & \qquad\qquad\qquad\quad 2 & \\
\cline{2-2}
4x^3 - 12x^2 + 11x - 3 & 12x^3 - 26x^2 + 18x - 4 & 3 \\
 & \underline{12x^3 - 36x^2 + 33x - 9} & \\
 & 5 \,|\, 10x^2 - 15x + 5 & \\
\end{array}
$$

$$
\begin{array}{r|l|l}
2x^2 - 3x + 1 & 4x^3 - 12x^2 + 11x - 3 & 2x - 3 \\
 & \underline{4x^3 - 6x^2 + 2x} & \\
 & \quad - 6x^2 + 9x - 3 & \\
 & \quad \underline{- 6x^2 + 9x - 3} & \\
\end{array}
$$

$\therefore\ 2x^2 - 3x + 1$ is the h. c. f.

Here the introduction of the factor 2 and the suppression of 5 evidently do not affect the *degree* of the highest common factor.

111. **The highest common factor of three expressions** cannot be of higher degree than that of any two; hence, the highest common factor of this highest common factor and of the third expression is the highest common factor of all three. Similarly, for any number of expressions.

EXERCISE XLV

Find the highest common factor of each of the following sets of expressions:

1. $x^3 - 2x + 4,\ x^4 + x^3 + 4x.$
2. $x^3 + 2x^2 + 2x + 1,\ x^3 - 1.$
3. $2x^3 + 2x - 4,\ x^3 - 3x + 2.$
4. $x^4 + 4,\ x^4 - 2x^3 + x^2 + 2x - 2.$
5. $x^6 + y^6,\ x^4 - y^4,\ x^5 + x^3y^2 + x^2y^3 + y^5.$

6. $x^3 - 40x + 63,\ x^4 - 7x^3 + 63x - 81.$

7. $x^4 + 2x^3 + x + 2,\ x^3 + 2x^2 + 2x + 4.$

8. $x^2(6x + 1) - x,\ 4x^3 - 2x(3x + 2) + 3.$

9. $x^3 - 8x^2 + 8x - 7,\ x^3 + 7x^2 - 7x + 8.$

10. $x^3 + 4x^2 + 2x - 1,\ x^3 + 5x^2 + 5x - 2.$

11. $x^3 - 2x^2 - 6x - 3,\ x^3 + x^2 - 15x - 12.$

12. $x^4 - 15x^2 + 28x - 12,\ 2x^3 - 15x + 14.$

13. $x^3 + 3x^2 + 7x + 5,\ x^3 + 5x^2 + 11x + 15.$

14. $x^3 + 9x^2 + 7x + 63,\ x^3 + 9x^2 + 9x + 81.$

15. $x^3 - 5x^2 - 7x + 35,\ x^3 - 3x^2 - 7x - 15.$

16. $x^3 - 5x^2 + 13x - 21,\ x^3 + 3x^2 - 3x + 35.$

17. $x^3 + 5x^2 - 7x + 49,\ x^3 + 7x^2 - 11x + 63.$

18. $x^3 - 7x^2 - 7x + 49,\ x^3 - 5x^2 - 11x - 21.$

19. $x^3 + 6x^2 + 6x + 36,\ x^3 + 8x^2 + 15x + 15.$

20. $x^2 - 4x - 117,\ x^4 - 13x^3 - x^2 + 14x - 13.$

21. $7x^3 - 10x^2 - 7x + 10,\ 2x^3 - x^2 - 2x + 1.$

22. $63a^4 - 17a^3 + 17a - 3,\ 98a^4 + 34a^2 + 18.$

23. $x^3 - 17x^2 + 5x - 85,\ x^3 - 17x^2 + 7x - 119.$

24. $x^2 + 4x - 21,\ x^2 + 20x + 91,\ 2x^3 + 4x^2 - 70x.$

25. $8x^4 - 10x^3 + 7x^2 - 2x,\ 6x^5 - 11x^4 + 8x^3 - 2x^2.$

26. $(a - b)(a^2 - c^2) - (a - c)(a^2 - b^2),\ a^5 - b^5,\ ab - b^2 - ac + bc.$

27. $x^3 - 10(x^2 + 3) + 31x,\ x^2(x - 11) + 2(19x - 20),\ x^3 - 9x^2 + 26x - 24.$

II. LOWEST COMMON MULTIPLE

112. The multiple of lowest degree common to two or more expressions is called their **lowest common multiple.**

E.g., a^3b^3cd is the lowest common multiple of a^3bc and ab^3d.

Similarly, $\pm(a+b)^2(a-b)$ is the lowest common multiple of a^2-b^2, $b-a$, and $(a+b)^2$. For

1. $$a^2-b^2=(a+b)(a-b).$$

2. $$b-a=-(a-b).$$

3. $$(a+b)^2=(a+b)(a+b).$$

4. $\therefore$ either $(a+b)^2(a-b)$ or $(a+b)^2(b-a)$ contains the given expressions and is the common multiple of lowest degree.

The lowest common multiple of algebra must not be considered the same as the least common multiple when numerical values are assigned. *E.g.*, the lowest common multiple of $a+b$ and $a-b$ is $(a+b)(a-b)$; but if $a=6$ and $b=4$, the least common multiple of $6+4$ and $6-4$ is simply $6+4$.

113. Factoring method. The lowest common multiple is usually found by the inspection of factors.

E.g., to find the lowest common multiple of x^2y and y^2z.

Since x^2, y^2, and z must appear,

$\therefore$ x^2y^2z is the lowest common multiple.

Similarly, to find the lowest common multiple of $x^2-12x+27$, x^2+x-12, and $15-2x-x^2$.

1. $$x^2-12x+27=\quad(x-3)(x-9).$$

2. $$x^2+x-12=\quad(x-3)(x+4).$$

3. $$15-2x-x^2=-(x-3)(x+5).$$

4. $\therefore$ $\pm(x-3)(x+4)(x+5)(x-9)$ is the lowest common multiple.

In practice, the result should be left in the factored form.

EXERCISE XLVI

Find the lowest common multiple of each of the following sets of expressions:

1. $abc^2,\ ab^2c,\ a^2bc.$
2. $a^4b^3,\ b^4c^3,\ c^4a^3.$
3. $x^2yz,\ z^2yx,\ y^2zx.$
4. $ax^2,\ by^2,\ abxy.$
5. $x^3 - xy^2,\ x^3 + xy^2.$
6. $ax^2y,\ bxy^2,\ cxyz.$
7. $mx^3,\ ny^3,\ m^2n^2x^2y^2.$
8. $x^2y^4z^6 - y^2,\ xy^2z^3 - y.$
9. $a(a + b),\ b(b + a).$
10. $a^2(x - y),\ b^2(x^2 - y^2).$
11. $a^2 + 2ab + b^2,\ a^2 - b^2.$
12. $a^2x^2 - b^2y^2,\ a^3x^3 - b^3y^3.$
13. $x^4 + 4,\ 2 - x^2,\ x^2 + 2.$
14. $pq^2,\ q(p + q),\ q(p - q).$
15. $a^3x^3 + 1,\ a^2x^2 - 1,\ ax + 1.$
16. $-10a^2xyz,\ 5x^2yz^3,\ a^3xy^3z.$
17. $m(a + b)(b + c),\ n(a + c).$
18. $ax - ay,\ bx^2 - by^2,\ cx + cy.$
19. $x^3 + y^3,\ x + y,\ xy - x^2 - y^2.$
20. $xa + xb,\ xa - xb,\ xa^2 - xb^2.$
21. $a(b - c),\ b(c - a),\ c(a - b).$
22. $a^2 + b^2 - 2ab,\ b^2 - a^2,\ a - b.$
23. $27 - 12x + x^2,\ x^2 + 2x - 15.$
24. $m^2 + mn + n^2,\ m^4 + n^4 + m^2n^2.$
25. $p(q + r),\ q(r + p),\ r(p + q).$
26. $ab^2c^3d^4,\ a^2b^3c^4d,\ a^3b^4cd^2,\ a^4bc^2d^3.$
27. $x(m + n)^2,\ y(m^2 - n^2),\ z(m - n)^2.$
28. $x^3 + y^3 + 3xy(x + y),\ x^3 + y^3,\ x + y.$
29. $x^2 + x - 12,\ -36 + 13x - x^2,\ x^2 - 16.$
30. $2xy - x^2 - y^2,\ 2xy + x^2 + y^2,\ x^2 - y^2,\ x + y.$

114. Highest common factor method. Since the highest common factor contains all of the factors common to *two* expressions, it may be suppressed from either of them and the quotient multiplied by the other to obtain the l.c.m.

For let $$x = af,$$ $$y = bf,$$

in which f is the highest common factor of x and y.

Then the lowest common multiple is evidently abf; *i.e.*, it is y multiplied by a.

E.g., to find the lowest common multiple of $2x^3 + 8x^2 - 3x - 27$ and $2x^3 + 12x^2 + x - 45$.

$$\begin{array}{l}
2x^3 + 12x^2 + x - 45 \\
\underline{2x^3 + 8x^2 - 3x - 27} \\
\quad 2 \,\underline{|\, 4x^2 + 4x - 18} \\
\quad\quad 2x^2 + 2x - 9 \,|\, 2x^3 + 8x^2 - 3x - 27 \,\underline{|\, x + 3} \\
\quad\quad\quad\quad\quad\quad \underline{2x^3 + 2x^2 - 9x} \\
\quad\quad\quad\quad\quad\quad\quad\quad 6x^2 + 6x - 27 \\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{6x^2 + 6x - 27}
\end{array}$$

$\therefore$ $(2x^3 + 12x^2 + x - 45)(x + 3)$ is the lowest common multiple.

In case of three or more expressions, it is better to factor all by the highest common factor, and then use the method given in § 113, thus avoiding any confusion.

EXERCISE XLVII

Find the lowest common multiple of each of the following sets of expressions:

1. $x^{12} + x^5$, $x^{17} + x^8$.
2. $x^4 + 3x^2 + 2$, $x^3 + 2x$.
3. $3a^3 - 11a^2 + 4$, $6a^2 - a - 2$.
4. $6x^2 + 5x + 1$, $12x^2 + 7x + 1$.
5. $x^6 + 3x^5 + x + 3$, $x^3 - 8x + 3$.

6. $6x^2 + 13x + 6,\ 9x^2 + 12x + 4.$

7. $x^3 + x^2 - 3x - 6,\ x^3 + 2x^2 - 3.$

8. $6x^2 + 13x + 6,\ 10x^2 - 3 + 13x.$

9. $x^2 + 5x + 6,\ x^3 + 3x^2 + 7x + 21.$

10. $x^4 - x^2 - 30,\ x^3 + 3x^2 - 6x - 18.$

11. $x^2 + 2ax + a^2,\ x^2 + ab + (a + b)x.$

12. $x^3 - x^2 - 3x + 3,\ x^3 + x^2 - 3x - 3.$

13. $x^3 + 2x^2 + 2x + 1,\ x^3 - x^2 - x - 2.$

14. $x^3 + 4x^2 + 3x - 2,\ x^3 + x^2 - 3x + 1.$

15. $6x^3 + 11x^2 - 9x + 1,\ 2x^2 + 3x - 2.$

16. $x^6 - 3x^4 - 2x^2 + 6,\ x^3 - x^2 - 3x + 3.$

17. $x^3 - x^2 + 3x - 3,\ x^3 - 5x^2 + 3x - 15.$

18. $x^4 - x^3 + x^2 - x - 4,\ x^4 - x^2 + 2x - 8.$

19. $x^3 - 10x^2 + 31x - 30,\ x^3 - 4x^2 + x + 6.$

20. $x^3 - 6x^2 - 7x - 6,\ x^3 - 9x^2 + 26x - 24.$

21. $x^3 + 5x^2 + 5x + 25,\ x^3 - 4x^2 + 5x - 20.$

22. $x^3 + 7x^2 + 5x + 35,\ x^3 - 6x^2 + 5x - 30.$

23. $x^3 + 1 + 3(x^2 + x),\ x^4 + 1 + 4(x^3 + x) + 6x^2.$

24. $3x^3 - 15ax^2 + a^2x - 5a^3,\ 6x^4 - 25a^2x^2 - 9a^4.$

25. $x^4 + x^3 + x + 1,\ 2x^5 - 3x^4 + 4x^3 + 2x^2 - 3x + 4.$

26. $x^2 + 20x + 91,\ 35 - 2x - x^2,\ x^4 + 6x^3 - 6x^2 + 6x - 7.$

27. $2x^4 - 2x^3 - x^2 - 4x - 7,\ 2x^4 + 6x^3 - 17x^2 + 8x - 35.$

28. $x^7 + x^6 - x^4 - 6x^2 - 6x - 7,\ x^8 - x^6 - x^5 + x^4 - 6x^3 - x + 7.$

29. $x^7 + 2x^6 - 3x^5 + x^2 + 2x - 3,\ x^7 + 4x^6 - 7x^5 + x^2 + 4x - 7,$
$x^3 + 1.$

REVIEW EXERCISE XLVIII

1. Multiply $(x+y)^2$ by $(x-y)^3$.

2. Factor the expression $4x^2y^2-(x^2+y^2-z^2)^2$.

3. Divide $a^2x^8+(2ac-b^2)x^4+c^2$ by ax^4-bx^2+c.

4. Factor the expression $amp-anq-bmp+bnq$.

5. Factor the expression $a^2bcd+ab^2cd+abc^2d+abcd^2$.

6. Factor the expression $0.125x^6-0.75x^4+1.5x^2-1$.

7. If $a=1$, $b=3$, $c=5$, $d=7$, find the value of $\dfrac{ab-cd}{ad-bc}$.

8. Simplify the expression $7x-\{3x-[4x-(5x-2x)]\}$.

9. From $(x+2)(x+3)(x+4)$ subtract $(x+1)(x+2)(x+3)$.

10. Factor the expression $a^2b^2c^3d^4+ab^4c^3d^4+ab^2c^6d^4+ab^2c^3d^8$.

11. Factor the expression $w^5x^2y^3z^4+wx^5y^3z^4+wx^2y^5z^4+wx^2y^3z^5$.

12. If $2s=x+y+z$, what does $(s-x)^2+(s-y)^2+(s-z)^2$ equal?

13. Divide the product of $x+y-z$, $x-y+z$, and $-x+y+z$ by $x^2-y^2-z^2+2yz$.

14. Find the highest common factor of $x^3-9x^2+26x-24$, $x^3-10x^2+31x-30$, and $x^3-11x^2+38x-40$.

15. Find the lowest common multiple of a^4-10a^2+9, $a^4+10a^3+20a^2-10a-21$, and $a^4+4a^3-22a^2-4a+21$.

16. Find the lowest common multiple of $4a^2(3a+2)-(27a+18)$, $12a^3-a(8a+27)+18$, and $6(3a-2)+27a^3-8$.

CHAPTER VII

FRACTIONS

115. The symbol $\frac{a}{b}$, in which b is not zero, is defined to mean the division of a by b, and is called an **algebraic fraction.**

Hence, the algebraic fraction $\frac{a}{b}$ represents a quantity which, when multiplied by b, produces a.

116. The **terms of the fraction** $\frac{a}{b}$ are a and b, a being called the **numerator** and b the **denominator,** and either or both may be fractional, negative, etc.

117. As in arithmetic we may cancel common factors from the terms, so we may do this in algebra, thus making the fraction easier to work with. One of the most important operations with algebraic fractions is this of *reduction to lower terms.*

E.g., in the fraction
$$\frac{a^2 - b^2}{(a + b)^2}$$
we have the factor $a + b$ common to both terms.

Hence
$$\frac{a^2 - b^2}{(a + b)^2} = \frac{(a + b)(a - b)}{(a + b)(a + b)} = \frac{a - b}{a + b},$$
and the fraction is reduced to lower (here the lowest) terms.

I. REDUCTION OF FRACTIONS

118. Theorem of reduction. *The same factor may be introduced into or cancelled from both numerator and denominator of a fraction without altering the value of the fraction.*

For by the definition of fraction

$$b \cdot \frac{a}{b} = a,$$

and hence $$nb \cdot \frac{a}{b} = na,$$

and hence $$\frac{a}{b} = \frac{na}{nb}, \qquad \text{by dividing by } nb.$$

Hence the terms of $\frac{a}{b}$ may be multiplied by n, or those of $\frac{na}{nb}$ divided by n, without altering the value of the fraction.

119. An algebraic fraction is said to be **simplified** when all common factors, and hence the highest common factor, of both numerator and denominator have been suppressed, and there is no fraction in either.

E.g., the fraction $\frac{a^2 + 2ab + b^2}{a^3 + b^3}$ is simplified when reduced to the form $\frac{a + b}{a^2 - ab + b^2}$ by cancelling the factor $a + b$.

But the fraction $\dfrac{\frac{a}{b} + b}{c}$ is not simplified.

120. The student should notice that the theorem does not allow the cancellation of any *terms* of the numerator and denominator. *Only factors can be cancelled.*

Usually the factors common to the two terms of the fraction can be found by inspection and cancelled; otherwise the highest common factor of both terms is cancelled.

ILLUSTRATIVE PROBLEMS

1. Simplify the fraction $\dfrac{-a^3b^2cd^4}{a^2b^3cd^3}$.

1. Cancelling a^2, b^2, c, and d^3, the fraction reduces to $\dfrac{-ad}{b}$.

2. And since there are no other common factors, and the terms are integral, the fraction is simplified.

2. Simplify $\dfrac{a^2+2\,ab+b^2}{a^2-b^2}$.

1. This evidently equals $\dfrac{(a+b)^2}{(a+b)(a-b)}$.

2. Cancelling $a+b$, this reduces to $\dfrac{a+b}{a-b}$.

3. And since there are no other common factors, and the terms are integral, the fraction is simplified.

Check. Let $a=2$, $b=1$. Then $\frac{9}{3}=\frac{3}{1}$. (If a and b are given the same values, the denominator becomes zero, a case excluded, for the present, by the definition of fraction.)

3. Simplify $\dfrac{3\,x^2+26\,x-77}{3\,x^2-10\,x+7}$.

1. A factor of each term of the fraction is a factor of their difference, $36\,x-84$ (§ 108, 2).

2. Hence of $3\,x-7$, because the terms of the fractions do not contain 12.

3. Hence, if there is a common factor, it is $3\,x-7$, because this is prime.

4. By trial this is seen to be a factor, and the fraction reduces by division to $\dfrac{x+11}{x-1}$.

Check. Let $x=2$. Then $\dfrac{-13}{-1}=\dfrac{13}{1}$.

4. Simplify $\dfrac{2\,x^3+9\,x^2+11\,x+14}{3\,x^3+4\,x^2+7\,x+2}$.

1. Here the simple factors are not as easily determined as the highest common factor, x^2+x+2.

2. Cancelling this, the fraction reduces to $\dfrac{2\,x+7}{3\,x+1}$.

3. $\therefore$ the fraction is, by definition, simplified.

Check. Let $x=1$. Then $\frac{36}{16}=\frac{9}{4}$.

121. General directions for simplifying fractions. The preceding fractions were simplified in different ways. While there is no general method of attack, and the student must use his judgment as to the best plan to pursue, the following directions are of value:

1. *Cancel monomial factors first,* as in Ex. 1.

2. *Then see if common polynomial factors can be readily discovered,* as in Ex. 2.

3. *If common factors are not readily discovered, see if the difference between the numerator and denominator can be easily factored.* If so, try these factors, as in Ex. 3.

4. *Let the method by finding the highest common factor be the final resort.* For one who is skilful in factoring, this tedious method ought rarely to be necessary.

EXERCISE XLIX

Simplify the following fractions:

1. $\dfrac{ab^2c^4}{bc^2}$.

2. $\dfrac{a^3-3a+2}{a^3+4a^2-5}$.

3. $\dfrac{a^2-b^2}{a^4-b^4}$.

4. $\dfrac{21x-10-9x^2}{3x^2-26x+35}$.

5. $\dfrac{x^3+y^3}{x^5+y^5}$.

6. $\dfrac{a^3+a^2+3a-5}{a^2-4a+3}$.

7. $\dfrac{x^4+x^2y}{x^4-y^2}$.

8. $\dfrac{6x^2+7xy-3y^2}{6x^2+11xy+3y^2}$.

9. $\dfrac{a^2bc^2d^3}{-ab^2cd^4}$.

10. $\dfrac{x^3-x^2-7x+3}{x^4+2x^3+2x-1}$.

11. $\dfrac{mx^2y-mxy^2}{nx^4y-nx^2y^3}$.

12. $\dfrac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2+2yz}$.

13. $\dfrac{a^3+a^2-2a}{a^3-a^2-6a}$.

14. $\dfrac{x^3+x^2-12x}{x^3+4x^2+5x+20}$.

15. $\dfrac{3x^2y^2+4xy^3}{5x^2y^2-4x^3y}$.

16. $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2}$.

17. $\dfrac{a^2+3a-10}{3a^2+2a-16}$.

18. $\dfrac{x^3-5x^2+7x-3}{2x^3-5x^2+4x-1}$.

19. $\dfrac{x^3+x^2y+xy^3}{x^5+x^3y^2+xy^4}$.

20. $\dfrac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}$.

21. $\dfrac{m^3-39m+70}{m^2-3m-70}$.

22. $\dfrac{m^3-6m^2+11m-6}{2m^3-14m+12}$.

23. $\dfrac{x^2-xy-12y^2}{x^2+5xy+6y^2}$.

24. $\dfrac{x^2+(m-n)x-mn}{x(x+m)-n(x+m)}$.

25. $\dfrac{x^2+(a+b)x+ab}{(x+a)(x+b)(x+c)}$.

26. $\dfrac{2a^2-10a-28}{3a^3-27a^2+21a+147}$.

27. $\dfrac{m^2x^2-(m+y)mnx+mn^2y}{x^3-(m+1)nx^2+mn^2x}$.

122. Reduction to integral or mixed expressions. Since the fraction $\dfrac{a}{b}$ indicates the division of a by b, it may be reduced to an integral form if the division is exact, and to a mixed form if the degree of the numerator equals or exceeds that of the denominator and the division is not exact.

E.g., $\dfrac{x^2-y^2}{x+y}=x-y$, the division being exact.

$\dfrac{x^2+y^2}{x+y}=x-y+\dfrac{2y^2}{x+y}$; that is, the division of the remainder by $x+y$ is indicated.

EXERCISE L

Reduce the following fractions to integral or mixed expressions:

1. $\dfrac{x^4+y^4}{x+y}$.

2. $\dfrac{x^3+2x}{x^2+1}$.

3. $\dfrac{a^5+3a^2-1}{a^5+1}$.

4. $\dfrac{x^2+7x+10}{x+2}$.

5. $\dfrac{x^2-9x+14}{x-2}$.

6. $\dfrac{12x^2+x+1}{4x-1}$.

7. $\dfrac{3x^2+2x-1}{3x-1}$.

8. $\dfrac{2x^2-3xy+y^2}{x+y}$.

9. $\dfrac{x^4+x^3-x-4}{x+1}$.

10. $\dfrac{4x^2+20x+21}{2x+3}$.

11. $\dfrac{4x^2-20x+21}{2x-3}$.

12. $\dfrac{x^4+2x^3+x-1}{x^2+2}$.

13. $\dfrac{x^3-x^2-3x+5}{x-1}$.

14. $\dfrac{x^6+2x^3+x^2-1}{x^3+2}$.

15. $\dfrac{12x^2-29x-12}{4x-3}$.

16. $\dfrac{x^3+y^3+z^3-3xyz}{x+y+z}$.

17. $\dfrac{x^3+6x^2+12x+8}{x+2}$.

18. $\dfrac{x^7+x^4+x^2-6}{x^6+x^5+x^3+x-6}$.

19. $\dfrac{4x^3+4x^2+2x+1}{2x^2+x+1}$.

20. $\dfrac{x^3+6x^2-6x-34}{x+6}$.

21. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}$.

22. $\dfrac{x^3+4x^2y+5xy^2+2y^3}{x^2+3xy+2y^2}$.

23. $\dfrac{2x^3-39x^2-38x+4}{x+1}$.

24. $\dfrac{x^5-2x^3+x^2+x-2}{x^2-2}$.

123. Reduction to equal fractions having the lowest common denominator.

In algebra as in arithmetic, it is often necessary to reduce fractions to equal fractions having a common denominator (preferably the lowest common denominator).

124. In algebra *we can reduce any fractions, like* $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, *to fractions having a common denominator* m *where* m *is a multiple of* b, d, f.

For $\because$ m is a multiple of b, d, f, we may let

$$m = pb,$$
$$m = qd,$$
$$m = rf.$$

And by § 118 and the above statements,

$$\frac{a}{b} = \frac{pa}{pb} = \frac{pa}{m},$$
$$\frac{c}{d} = \frac{qc}{qd} = \frac{qc}{m},$$
$$\frac{e}{f} = \frac{re}{rf} = \frac{re}{m},$$

all three fractions having the common denominator m.

125. In particular, if m is the lowest common multiple of the denominators, the fractions will be reduced to equal fractions having the lowest common denominator.

E.g., to reduce the fractions $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$ to equal fractions having the lowest common denominator:

$$\frac{x+y}{x-y} = \frac{(x+y)^2}{(x+y)(x-y)}.$$
$$\frac{x-y}{x+y} = \frac{(x-y)^2}{(x+y)(x-y)}.$$

EXERCISE LI

Reduce the following to equal fractions having the lowest common denominator:

1. $\frac{a}{b}, \frac{b}{a}$.

2. $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$.

3. $\frac{x}{yz}, \frac{y}{zx}, \frac{z}{xy}$.

4. $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}$.

5. $\frac{ab}{c^2d}, \frac{-b}{c^3d^2}, \frac{a^2b}{de^4}$.

6. $\frac{a}{b+c}, \frac{b}{c+a}$.

7. $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$.

8. $\frac{ax}{a+x}, \frac{by}{b+y}$.

9. $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$.

10. $\frac{xy}{x^2+y^2}, \frac{x^2y^2}{x^4-y^4}, \frac{x}{x-y}$

11. $\frac{x+1}{x^2+4x+3}, \frac{x-1}{x^2-9}, \frac{x}{x-3}$.

12. $\frac{1}{m^2+6m+8}, \frac{2}{2m^2+7m+6}$.

13. $\frac{2m-2n}{m^2-mn+n^2}, \frac{4(m+n)}{5(m^2+mn+n^2)}$.

14. $\frac{a^2-b^2}{a^3-b^3}, \frac{(a+b)^2}{a^3+b^3}, \frac{a^2+b^2+2ab}{(a^2-b^2)^2}$.

15. $\frac{9x^2+12xy-5y^2}{3x^2-xy-10y^2}, \frac{6x^2-11xy+4y^2}{2x^2-5xy+2y^2}$.

16. $\frac{x-y}{x^2-y(x-y)}, \frac{x+y}{x^2+y(x+y)}, \frac{2x^3y^3}{x^6-y^6}$.

17. $\frac{x+1}{x^2+5x+6}, \frac{x+2}{x^2+4x+3}, \frac{x+3}{x^2+3x+2}$.

18. $\frac{a-3}{a^2-9a+18}, \frac{2a+8}{a^2+a-12}, \frac{a+5}{a^2+8a+15}$.

19. $\frac{xy}{(y+z)(z+x)}, \frac{yz}{x^2+zy+zx+xy}, \frac{zx}{y^2+yz+xy+xz}$.

II. ADDITION AND SUBTRACTION

126. It has already been proved, § 73, that

$$\frac{a}{k}+\frac{b}{k}+\frac{c}{k}=\frac{a+b+c}{k}.$$

Hence, *if fractions have a common denominator* k, *their sum is the sum of their numerators divided by the common denominator.*

For simplicity it is, of course, better to reduce to equal fractions having the *lowest* common denominator.

Thus, with numerical fractions,

$$\tfrac{2}{3}+\tfrac{5}{6}=\tfrac{4}{6}+\tfrac{5}{6}=\tfrac{9}{6}=\tfrac{3}{2}.$$

And similarly with algebraic fractions,

$$\frac{a}{bc}+\frac{b}{cd}=\frac{ad}{bcd}+\frac{b^2}{bcd}=\frac{ad+b^2}{bcd}.$$

ILLUSTRATIVE PROBLEMS

1. Required the sum of $\dfrac{a}{b-c}$, $\dfrac{a}{b+c}$.

1. The l. c. m. of the denominators is $(b+c)(b-c)$.

2. $$\frac{a}{b-c}=\frac{(b+c)a}{(b+c)(b-c)}.$$

3. $$\frac{a}{b+c}=\frac{(b-c)a}{(b+c)(b-c)}.$$

4. $$\therefore\quad \frac{a}{b-c}+\frac{a}{b+c}=\frac{(b+c)a+(b-c)a}{(b+c)(b-c)}$$
$$=\frac{2ab}{(b+c)(b-c)}.$$

Check. If $a=1$, $b=2$, $c=1$, then $\frac{1}{1}+\frac{1}{3}=\frac{4}{3}$.

2. Simplify the polynomial $\dfrac{x}{x^2-1}+\dfrac{x+3}{x-1}-\dfrac{x-2}{x+1}$.

1. The l. c. m. of the denominators is x^2-1.

2. $$\frac{x+3}{x-1}=\frac{(x+1)(x+3)}{x^2-1}.$$

3. $$\frac{x-2}{x+1}=\frac{(x-1)(x-2)}{x^2-1}.$$

4. $$\therefore \frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-2}{x+1}=\frac{x+(x+1)(x+3)-(x-1)(x-2)}{x^2-1}$$
$$=\frac{x+x^2+4x+3-x^2+3x-2}{x^2-1}$$
$$=\frac{8x+1}{x^2-1}.$$

Check. Let $x=2$. Then $\frac{2}{3}+\frac{5}{1}-\frac{0}{3}=\frac{17}{3}$.

127. In a case like $\dfrac{x+y}{x^2+y^2}-\dfrac{x-y}{x^2+y^2}$ it must be remembered that *the bar separating numerator and denominator is a sign of aggregation.*

In this case the result is $\dfrac{x+y-(x-y)}{x^2+y^2}=\dfrac{x+y-x+y}{x^2+y^2}=\dfrac{2y}{x^2+y^2}$.

EXERCISE LII

Simplify the following expressions:

1. $\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}$.

2. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

3. $\dfrac{2x}{3yz^2}+\dfrac{5z}{6y^2w}$.

4. $\dfrac{2-x}{1-x^2}+\dfrac{x-2}{1-x-2x^2}$.

5. $\dfrac{1}{x+y}+\dfrac{1}{x-y}$.

6. $\dfrac{x}{x+y}+\dfrac{x}{x-y}-\dfrac{2xy}{x^2-y^2}$.

7. $\dfrac{a+1}{a+2}-\dfrac{a+2}{a+3}+\dfrac{a-1}{a+2}$.

8. $\dfrac{(x+y)^2}{(x-y)^2}+\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}$.

9. $\dfrac{m}{xy}-\dfrac{n}{yz}.$

10. $\dfrac{x}{3\,yz^2}+\dfrac{y}{3\,xz^2}+\dfrac{z}{3\,xy^2}.$

11. $\dfrac{m}{p^2q}+\dfrac{n}{pq^2}.$

12. $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}.$

13. $\dfrac{a-b}{mc}+\dfrac{b}{nc}.$

14. $\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}.$

15. $\dfrac{1}{x+1}+\dfrac{1}{x-1}.$

16. $\dfrac{x+2\,y}{x^2-y^2}-\dfrac{x-2\,y}{x+y}-\dfrac{x}{x-y}.$

17. $\dfrac{q+r}{p^2+q}+\dfrac{p+q}{q^2+r}.$

18. $\dfrac{x}{x^3-1}+\dfrac{x}{x^3+1}-\dfrac{x-1}{x+1}.$

19. $\dfrac{x^2-x-1}{x^6-1}+\dfrac{1}{x-1}.$

20. $\dfrac{x+1}{x^2+x+1}+\dfrac{x-1}{x^2-x+1}.$

21. $\dfrac{5\,x^4-7\,x^3-9\,x^2+11}{2\,x^4-3\,x^3+2\,x^2-1}-\dfrac{x-1}{x+3}.$

22. $\dfrac{a-4}{a^2-9\,a+20}+\dfrac{a-5}{a^2-11\,a}+\dfrac{a-3}{a^2-7\,a+12}.$

23. $\dfrac{1}{1+x}-\dfrac{2(1-x)}{(1+x)^2}+\dfrac{1+x^2}{(1+x)^3}-\dfrac{6\,x^2(1-x)}{(1+x)^4}.$

24. $\dfrac{a^2+ab+b^2}{a+b}-\dfrac{a^2-ab+b^2}{a-b}+\dfrac{2\,b^3-b^2+a^2}{a^2-b^2}.$

25. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}.$

26. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}.$

27. $\dfrac{xy}{(y-z)(z-x)}+\dfrac{yz}{(z-x)(x-y)}+\dfrac{zx}{(x-y)(y-z)}.$

28. $\dfrac{ax^2+byz}{(x-y)(x-z)}+\dfrac{ay^2+bzx}{(y-z)(y-x)}+\dfrac{az^2+bxy}{(z-x)(z-y)}.$

29. $\dfrac{x}{y}+\dfrac{2\,x^2+y^2}{xy}+\dfrac{3\,xy^2-3\,x^3-y^3}{x^2y}-\dfrac{4\,xy^3-2\,x^2y^2-y^4}{x^2y^2}.$

30. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}+\dfrac{1}{c(c-a)(c-b)}.$

128. Reduction of integral or mixed expressions to fractional form.

An integer can always be expressed as a fraction with any denominator.

Thus to express 3 as a certain number of 5ths. Since

$$1=\frac{5}{5},\quad 3=\frac{3\cdot 5}{5}=\frac{15}{5}.$$

Similarly, to express a as a certain number of $b$ths. Since

$$1=\frac{b}{b},\quad a=\frac{ab}{b}.$$

129. A mixed expression can also be expressed as a fraction.

For since $\qquad a+\dfrac{b}{c}=\dfrac{ac}{c}+\dfrac{b}{c},\qquad$ § 118

therefore $\qquad a+\dfrac{b}{c}=\dfrac{ac+b}{c}.\qquad$ § 126

ILLUSTRATIVE PROBLEMS

1. Express $a-b$ as a fraction with denominator $a+b$.

Since $\qquad 1=\dfrac{a+b}{a+b},$

therefore $\qquad a-b=\dfrac{(a-b)(a+b)}{a+b}=\dfrac{a^2-b^2}{a+b}.$

2. Express $x-\dfrac{x^2-y}{x+y}$ as a fraction.

$$x-\frac{x^2-y}{x+y}=\frac{x(x+y)}{x+y}-\frac{x^2-y}{x+y}$$
$$=\frac{x^2+xy-x^2+y}{x+y}$$
$$=\frac{xy+y}{x+y}.$$

Check. Let $x=y=1$; then $1-0=\dfrac{1+1}{1+1}.$

EXERCISE LIII

Write the expressions in Exs. 1–11 as fractions with the denominators indicated, as in § 128.

1. 5,	denominator	$25a$.
2. ab^2,	"	$a-b$.
3. abc,	"	abc.
4. $x+y$,	"	$x-y$.
5. $ab^2c^3d^4$,	"	$a^4b^3c^2d$.
6. a^3-b^3,	"	a^3+b^3.
7. $3a+4$,	"	$4a+3$.
8. a^2+b+c,	"	$a+c$.
9. x^2+2x+1,	"	$x+1$.
10. x^3+x^2+x+1,	"	$x-1$.
11. $x^4-x^3+x^2-x+1$,	"	$x+1$.

Reduce the following expressions to fractional forms:

12. $a+\dfrac{a}{b}$.

13. $a^2-b^2-\dfrac{2b^4}{a^2+b^2}$.

14. $a^2b-\dfrac{ab^2}{a-b}$.

15. $4x^2+1+\dfrac{2}{3x-2}$.

16. $a-b+\dfrac{1}{a+b}$.

17. $x^2-x+1-\dfrac{1}{x+1}$.

18. $4a-\dfrac{6ab-2}{3b}$.

19. $x^2+x+1+\dfrac{2}{x-1}$.

20. $a+b+\dfrac{b^2}{a-b}$.

21. $x^3-3x-\dfrac{3x(3-x)}{x-2}$.

22. $x+1+\dfrac{2}{x-1}$.

23. $3x+2y-\dfrac{6y^2}{5x+3y}$.

24. $m^2+1+\dfrac{1}{m-1}$.

25. $2x+5y-\dfrac{10y^2}{5x+2y}$.

III. MULTIPLICATION

130. Fractions are multiplied in algebra in the same way as in arithmetic, by taking *the product of the numerators for a numerator, and the product of the denominators for a denominator.*

This is easily proved. Let $\frac{a}{b}$, $\frac{c}{d}$ be two fractions, and let x be their product. We are to show that $x = \frac{ac}{bd}$.

Since $\qquad x = \frac{a}{b} \cdot \frac{c}{d}$,

therefore $\qquad bdx = a \cdot c$, by multiplying by b and d.

Therefore $\qquad x = \frac{ac}{bd}$, by dividing by bd.

ILLUSTRATIVE PROBLEMS

1. Find the product of $\frac{a+b}{a-b}$ and $\frac{a^2-b^2}{a^3+b^3}$.

$$\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{a^3+b^3} = \frac{(a+b)(a+b)(a-b)}{(a-b)(a+b)(a^2-ab+b^2)}$$

$$= \frac{a+b}{a^2-ab+b^2}. \qquad \text{§ 118}$$

Check. Let $a = 2$, $b = 1$; then $\frac{3}{1} \cdot \frac{3}{9} = \frac{3}{3}$.

131. The student should notice the advantage of factoring the expressions. In general it is better *never to multiply until compelled to, always to factor if possible.*

2. Find the product of $\frac{x^2 - 8x + 15}{x^2 - 12x + 35}$ and $\frac{x^2 - 15x + 56}{x^2 - 17x + 72}$.

Factoring, the product is

$$\frac{(x-5)(x-3)(x-7)(x-8)}{(x-5)(x-7)(x-9)(x-8)} = \frac{x-3}{x-9}.$$

Check. Let $x = 1$; then $\frac{8}{24} \cdot \frac{42}{56} = \frac{-2}{-8}$.

3. Find the product of $a+b$ and $\frac{a}{a^2-b^2}$. Since $a+b = \frac{a+b}{1}$, this is merely a special case under § 130. Therefore

$$\frac{a+b}{1}\cdot\frac{a}{a^2-b^2}=\frac{(a+b)a}{(a+b)(a-b)}=\frac{a}{a-b}.$$

4. To find the product of $\frac{x-a}{x+a}$, $\frac{x-2a}{x+2a}$, and $\frac{x}{x-a}$.

1. $$\frac{x-a}{x+a}\cdot\frac{x-2a}{x+2a}\cdot\frac{x}{x-a}=\frac{(x-a)(x-2a)x}{(x+a)(x+2a)(x-a)}$$

2. $$=\frac{(x-2a)x}{(x+a)(x+2a)}.$$ § 118

Check. Let $x=3$, $a=1$. Then $\frac{2}{4}\cdot\frac{1}{5}\cdot\frac{3}{2}=\frac{3}{4\cdot 5}$.

EXERCISE LIV

Perform the multiplications indicated, simplifying the results as usual.

1. $\frac{ab^2c^3}{pq^2r^3}\cdot\frac{p^3q^2r}{a^3b^2c}$.

2. $\frac{x^2y}{a^2b}\cdot\frac{y^2z}{b^2c}\cdot\frac{abc}{xyz}$.

3. $\frac{ab^2xy^2}{a^3bx^3y}\cdot\frac{a^2bx^2y}{ab^3xy^3}$.

4. $\frac{abc}{def}\cdot\frac{bcd}{efa}\cdot\frac{cde}{fab}$.

5. $\frac{a^2b}{a+b}\cdot\frac{a^2-b^2}{ab^2}$.

6. $\frac{a+b}{a-b}\cdot\frac{a^2-b^2}{a^2+b^2}$.

7. $\frac{7x^3y^2}{12x^2y^3}\cdot\frac{18x^4y^5}{28x^5y^4}$.

8. $\frac{a^3-b^3}{a+b}\cdot\frac{a^3+b^3}{a-b}$.

9. $\frac{x^6-y^6}{x^3-y^3}\cdot\frac{x^2-y^2}{(x+y)^2}$.

10. $\frac{27x}{8y+8x}\cdot\frac{x+y}{3}$.

11. $\frac{p^2qr}{3pq^2r^3}\cdot\frac{pq^2r}{4p^4}\cdot\frac{36p^3r}{q}$.

12. $\frac{a+1}{a-1}\cdot\frac{a+2}{a-2}\cdot\frac{a+3}{a-3}$.

13. $\frac{x^3-y^3}{x^3+y^3}\cdot\frac{x^2-xy+y^2}{x^2+xy+y^2}$.

14. $\frac{x^2+3x+2}{x^2+4x+3}\cdot\frac{x+3}{x+2}$.

15. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}\cdot\dfrac{a^3-b^3}{a^3+b^3}.$ 16. $\dfrac{a^2+b^2+2\,ab}{a-b}\cdot\dfrac{1}{a^2-b^2}.$

17. $\dfrac{(a+b)(x+y)}{(a-b)(x-y)}\cdot\dfrac{a^2-b^2}{x+y}.$ 18. $\dfrac{x+3}{x^2+3\,x+2}\cdot\dfrac{x+2}{x^2+4\,x+3}.$

19. $\dfrac{x^4-y^4}{(x-y)^2}\cdot\dfrac{x-y}{x^2+xy}\cdot\dfrac{x}{x^2+y^2}.$

20. $\dfrac{x^4+y^4+x^2y^2}{x^2+y^2+2\,xy}\cdot\dfrac{x+y}{x^2+y^2+xy}.$

21. $\dfrac{x^4+x^3y+xy^3+y^4}{x^2+2\,xy+y^2}\cdot\dfrac{x-y}{x^2+xy}.$

22. $\dfrac{x^2+x-12}{x^2-13\,x+40}\cdot\dfrac{x^2+2\,x-35}{x^2+9\,x+20}.$

23. $\dfrac{x^2+5\,x+6}{x^2+7\,x+12}\cdot\dfrac{x^2+9\,x+20}{x^2+11\,x+30}.$

24. $\dfrac{2\,a^2+5\,a+2}{6\,a^2+5\,a+1}\cdot\dfrac{9\,a^2+15\,a+4}{5\,a^2+12\,a+4}.$

25. $\dfrac{4\,x^2-6\,x+2}{4\,x^2-8\,x+3}\cdot\dfrac{4\,x^2-10\,x+6}{4(x^2-2\,x+1)}.$

26. $\dfrac{x^2+15\,x+56}{x^2+11\,x+30}\cdot\dfrac{x+6}{x^2+12\,x+35}.$

27. $\dfrac{10\,x^2+17\,x+3}{4\,x-1}\cdot\dfrac{20\,x^2-x-1}{2\,x+3}.$

28. $\dfrac{12\,x^2+5\,x-2}{6\,x^2+13\,x+6}\cdot\dfrac{8\,x^2+10\,x-3}{4\,x-1}.$

29. $\dfrac{8\,a^3+1}{2\,a-1}\cdot\dfrac{8\,a^3-1}{4\,a^2+2\,a+1}\cdot\dfrac{a}{2\,a+1}.$

30. $\dfrac{x^4+16\,m^4+4\,m^2x^2}{x^2-16}\cdot\dfrac{x+4}{x^2+4\,m^2-2\,mx}.$

31. $\dfrac{a^2-b^2}{a^3+b^3}\cdot\dfrac{a^3-b^3}{a^2+b^2}\cdot\dfrac{a+b}{a-b}\cdot\dfrac{a^4+a^2b^2}{a^2+ab+b^2}.$

IV. POWERS OF FRACTIONS

132. It is evident, from the laws of multiplication, that the square of $\frac{a}{b}$ is $\frac{a}{b}\cdot\frac{a}{b}=\frac{a^2}{b^2}$. Similarly it is evident that the cube of $\frac{a}{b}$ is $\frac{a^3}{b^3}$. This law is general; for

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots \text{ to } n \text{ factors}$$

$$=\frac{aaa\cdots \text{ to } n \text{ factors}}{bbb\cdots \text{ to } n \text{ factors}}$$

$$=\frac{a^n}{b^n}.$$

133. *Therefore any integral power of a fraction equals that power of the numerator divided by that power of the denominator.*

ILLUSTRATIVE PROBLEMS

1. $$\left(\frac{a+b}{a-b}\right)^2=\frac{a^2+2ab+b^2}{a^2-2ab+b^2}.$$

2. To express $\frac{a^2b^2}{c^4d^6}$ as the power of a fraction. Evidently

$$\frac{a^2b^2}{c^4d^6}=\left(\frac{ab}{c^2d^3}\right)^2.$$

3. To express $\frac{x^2+4x+4}{x^2-6x+9}$ as the power of a fraction.

$$\frac{x^2+4x+4}{x^2-6x+9}=\frac{(x+2)^2}{(x-3)^2}$$

$$=\left(\frac{x+2}{x-3}\right)^2.$$

4. To find the product of $\frac{a}{b}+\frac{b}{a}$ and $\frac{a}{b}-\frac{b}{a}$.

1. $$\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a}{b}-\frac{b}{a}\right)=\left(\frac{a}{b}\right)^2-\left(\frac{b}{a}\right)^2$$ § 68

2. $$=\frac{a^2}{b^2}-\frac{b^2}{a^2}$$ § 133

3. $$=\frac{a^4-b^4}{a^2b^2}.$$ §§ 125, 126

Check. Let $a=1$, $b=2$. Then

$$\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-2\right)=\frac{1-16}{4},\ \text{for}\ \frac{5}{2}\cdot\frac{-3}{2}=\frac{-15}{4}.$$

5. To find the product of $\frac{x^2+6x+5}{x^2+7x+12}\cdot\frac{x^2+8x+15}{x^2+5x+4}$.

1. $$\frac{x^2+6x+5}{x^2+7x+12}\cdot\frac{x^2+8x+15}{x^2+5x+4}$$

2. $$=\frac{(x+1)(x+5)(x+3)(x+5)}{(x+3)(x+4)(x+1)(x+4)}$$ § 130

3. $$=\frac{(x+5)^2}{(x+4)^2}.$$ § 118

Check. $\frac{12}{20}\cdot\frac{24}{10}=\frac{36}{25}$.

EXERCISE LV

Write the expressions in Exs. 1–12 without using the parentheses.

1. $\frac{(a-b)^2}{(a+b)^2}$. 2. $\left(\frac{x+3}{x-3}\right)^2$. 3. $\frac{m^2+m+1}{(m+1)^2}$.

4. $\left(\frac{a+b}{abc}\right)^2$. 5. $\left(\frac{p+q}{p-r}\right)^2$. 6. $\frac{a^2-ab+b^2}{(a-b)^2}$.

7. $\left(\dfrac{abc}{a-b}\right)^3.$ 8. $\left(\dfrac{x^3+1}{x^3-1}\right)^2.$ 9. $\left(\dfrac{ax^2+by^2}{ax^2-by^2}\right)^2.$

10. $\left(\dfrac{x+y}{x-y}\right)^3.$ 11. $\left(\dfrac{x^2y^2z^2}{x^2+y^2}\right)^2.$ 12. $\dfrac{(2x-3)^2}{9+4x^2-12x}.$

Write the expressions in Exs. 13–32 as powers of a fraction:

13. $\dfrac{p^4+2p^2+1}{p^4-2p^2+1}.$

14. $\dfrac{121+22x+x^2}{121-22x+x^2}.$

15. $\dfrac{4a^2+4ab+b^2}{b^2+9a^2-6ab}.$

16. $\dfrac{4x^2-4xy+y^2}{4y^2-4xy+x^2}.$

17. $\dfrac{x^2+169-26x}{x^4+1+2x^2}.$

18. $\dfrac{x^2y^2z^2+2xyz+1}{x^2y^2z^2-2xyz+1}.$

19. $\dfrac{a^2b^2+4ab+4}{b^4c^4+6b^2c^2+9}.$

20. $\dfrac{4x^2-12xy+9y^2}{9x^2-12xy+4y^2}.$

21. $\dfrac{x^2+6xy^2+9y^4}{y^2+6x^2y+9x^4}.$

22. $\dfrac{a^2x^2-2abxy+b^2y^2}{b^2x^2-2abxy+a^2y^2}.$

23. $\dfrac{x^2-8xy+16y^2}{16x^2-8xy+y^2}.$

24. $\dfrac{a^4+9b^4+6a^2b^2}{81(a^2+b^2)+162ab}.$

25. $\dfrac{a^2+10ab+25b^2}{1+4x+4x^2}.$

26. $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^3+3x^2+3x+1}.$

27. $\dfrac{100x^4+20x^2+1}{x^4+20x^2+100}.$

28. $\dfrac{x^3-9x^2+27x-27}{27x^3-27x^2+9x-1}.$

29. $\dfrac{4x^2+9y^2+12xy}{4x^2+9y^2-12xy}.$

30. $\dfrac{8a^3-36a^2+54a-27}{27a^3-54a^2+36a-8}.$

31. $\dfrac{9x^2-24xy+16y^2}{16x^2-24xy+9y^2}.$

32. $\dfrac{a^2b^2x^2+2ab^2cxy+b^2c^2y^2}{a^2b^2x^2-2ab^2cxy+b^2c^2y^2}.$

Perform the multiplications indicated in Exs. 33–50, and simplify each result.

33. $\left(\frac{2x}{3}-1\right)^3$.

34. $\left(\frac{a}{b}+\frac{b}{a}\right)^2$.

35. $\left(1+\frac{x}{y}\right)\left(1-\frac{x}{y}\right)$.

36. $\left(\frac{a}{b}\right)^{97}\cdot\left(\frac{b}{c}\right)^{99}\cdot\left(\frac{c}{a}\right)^{98}$.

37. $\left(\frac{m^2}{n}+\frac{n^2}{m}\right)^2$.

38. $\frac{a+b}{a-b}\left(\frac{1}{b}-\frac{1}{a}\right)$.

39. $\left(\frac{1}{b}+\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right)$.

40. $\left(\frac{a^2}{a+b}\right)^2\cdot\left(\frac{a+b}{a^2}\right)^3$.

41. $\frac{x+y}{x-y}\left(x+\frac{y^2}{x-y}\right)$.

42. $\frac{ac+bc+a^2+ab}{ab+ac+b^2+bc}\cdot\frac{b+c}{c+a}$.

43. $\frac{ab-ac-b^2+bc}{bc-ab-c^2+ac}\cdot\frac{c-a}{a-b}$.

44. $\frac{3}{xy}\cdot\left(x+\frac{xy}{3}\right)\cdot\left(1-\frac{3}{3+y}\right)$.

45. $\frac{4a^2+12a+9}{3a+2}\cdot\frac{9a^2+12a+4}{2a+3}$.

46. $\left(\frac{x^2+1}{x+1}\right)^2\cdot\left(\frac{x^2-1}{x+1}\right)^2\cdot\left(\frac{x^2+1}{x-1}\right)^2$.

47. $\left[\frac{1}{3}+\frac{2x}{3(1-x)}\right]\cdot\left[\frac{3}{4}-\frac{3x}{2(1+x)}\right]$.

48. $1-\frac{a+b}{a-b}\left(\frac{a}{a+b}-\frac{a-b}{a}+\frac{a-b}{a+b}\right)$.

49. $\left(\frac{p}{a}-\frac{q}{b}\right)\frac{r}{c}+\left(\frac{p}{a}-\frac{r}{c}\right)\frac{q}{b}+\left(\frac{q}{b}-\frac{r}{c}\right)\frac{p}{a}$.

50. $\left(\frac{1}{a}+\frac{1}{b}+\frac{a+b}{(a-b)^2}\right)\cdot\left(\frac{a^2+ab}{a^2-ab+b^2}\right)$.

V. DIVISION

134. The fraction formed by interchanging the numerator and denominator of a fraction (of which neither term is zero) is called the **reciprocal** of that fraction.

E.g., 2 is the reciprocal of $\frac{1}{2}$, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

Evidently 1 and -1 are the only numbers which are their own reciprocals respectively.

The term *reciprocal* is used only in relation to abstract numbers.

135. *Division of fractions in algebra is performed as in arithmetic, by multiplying by the reciprocal of the divisor.*

To prove this fact let us show that $q \div \frac{a}{b}$ equals $q \times \frac{b}{a}$.

Let $$q \div \frac{a}{b} = x;$$

then $$q = \frac{a}{b} \cdot x, \text{ by multiplying by } \frac{a}{b}.$$

Multiplying by $\frac{b}{a}$,

$$\frac{b}{a} \cdot q = \frac{b}{a} \cdot \frac{a}{b} \cdot x = x, \quad \because \frac{b}{a} \cdot \frac{a}{b} = 1.$$

That is, $$\frac{b}{a} \cdot q = q \div \frac{a}{b}.$$

ILLUSTRATIVE PROBLEMS

1. Perform the following division:

$$\frac{27}{8(x^2 - y^2)} \div \frac{3x}{x - y}.$$

$$\frac{27}{8(x^2-y^2)} \div \frac{3x}{x-y} = \frac{(x-y)27}{3x \cdot 8(x+y)(x-y)} \qquad \text{§ 135}$$

$$= \frac{9}{8x(x+y)}, \text{ cancelling } 3(x-y). \qquad \text{§ 118}$$

2. Perform the following division: $\dfrac{x^3-a^3}{x+a} \div \dfrac{x-a}{x^3+a^3}$.

$$\frac{x^3-a^3}{x+a} \div \frac{x-a}{x^3+a^3} = \frac{(x^3+a^3)(x^3-a^3)}{(x-a)(x+a)} \qquad \S\ 135$$

$$= (x^2 - xa + a^2)(x^2 + xa + a^2).$$

3. Perform the following division:

$$\left(\frac{a}{a-b} - \frac{b}{a+b}\right) \div \frac{a^2+b^2}{a^2-ab}.$$

1. $$\frac{a}{a-b} - \frac{b}{a+b} = \frac{a^2+b^2}{(a-b)(a+b)}.$$

2. $$\frac{a^2+b^2}{(a-b)(a+b)} \div \frac{a^2+b^2}{a^2-ab} = \frac{a^2+b^2}{(a-b)(a+b)} \cdot \frac{a(a-b)}{a^2+b^2}$$

$$= \frac{a}{a+b}.$$

EXERCISE LVI

Perform the following divisions, simplifying each result:

1. $\dfrac{2\,a^2b}{3\,b^2c} \div \dfrac{ab^2}{6\,b^3c}$.

2. $\dfrac{a^2b^3c^4}{x^3y^2z} \div \dfrac{a^4b^3c^3}{xy^2z}$.

3. $\dfrac{abx^3y}{cdxy^3} \div \dfrac{a^2b^2xy^3}{c^2d^2x^3y}$.

4. $\dfrac{a^3+b^3}{a^3-b^3} \div \dfrac{a+b}{a-b}$.

5. $\dfrac{a^2+ab}{b^2+ba} \div \dfrac{c^2+cd}{d^2+dc}$.

6. $\left(a+\dfrac{b}{c}\right) \div \left(\dfrac{a}{c}+\dfrac{b}{c^2}\right)$.

7. $\dfrac{p^3q^3r^3}{(x+y)^2} \div \dfrac{(x+y)^2}{p^3q^3r^3}$.

8. $\left(2\,x-\dfrac{x}{y}\right) \div (2\,y-1)$.

9. $\dfrac{x^2+x(a+b)+ab}{x^2+x(b+c)+bc} \div \left(\dfrac{x+a}{x+c}\right)^2$.

10. $\dfrac{a^2+2\,a-15}{a^2+8\,a-33} \div \dfrac{a^2+9\,a+20}{a^2+7\,a-44}$.

11. $\dfrac{a^2b^2m^2 - b^2c^2n^2}{a^3m + a^2n} \div \dfrac{ab^2m - b^2cn}{a^2m^2 - n^2}$.

12. $\left[x + \dfrac{b(y - x)}{a + b}\right] \div \dfrac{ax - by}{a^2 + 2ab + b^2}$?

13. $\dfrac{a^2 + b^2 - c^2 + 2ab}{(a + b + c)^2} \div \dfrac{(a + b - c)^2}{abc}$.

14. $\left(\dfrac{a}{1 + a} + \dfrac{a}{1 - a}\right) \div \left(\dfrac{a}{1 + a} - \dfrac{1 - a}{a}\right)$.

15. $\left(\dfrac{x}{x + 1} - \dfrac{x - 1}{x}\right) \div \left(\dfrac{x}{x + 1} + \dfrac{x - 1}{x}\right)$.

16. $\left(\dfrac{6}{5 - 4x} - \dfrac{14}{2 - x}\right) \div \dfrac{25x - 29}{(5 - 4x)(2 - x)}$.

17. $\left(\dfrac{a^2 + b^2}{a^2 - b^2} - \dfrac{a^2 - b^2}{a^2 + b^2}\right) \div \left(\dfrac{a + b}{a - b} - \dfrac{a - b}{a + b}\right)$.

18. $\left(\dfrac{7a - 13b}{a - 3b} + \dfrac{2a - 5b}{3b - a} - 2\tfrac{2}{3}\right) \div \dfrac{1}{a - 3b}$.

19. $\dfrac{x^3 + 6x^2 + 12x + 8}{x^2 - 4x + 4} \div \dfrac{x^2 + 4x + 4}{x^3 - 6x^2 + 12x - 8}$.

20. $\dfrac{x^2 - 6xy + 9y^2}{x^2 - 4xy + 4y^2} \div \left(\dfrac{x^2 - 9y^2}{x^2 + 4y^2} \div \dfrac{x^2 + xy - 6y^2}{x^2 - xy - 6y^2}\right)$.

21. $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{a^2 - 2ab + b^2} \div \dfrac{a^2 + 2ab + b^2}{a^3 - 3a^2b + 3ab^2 - b^3}$.

22. $\dfrac{x^2 - a^2 + 4y^2 - 4b^2 + 4xy + 4ab}{x^2 + 6ax + 9a^2} \div \dfrac{x + a + 2(y - b)}{x^2 - 9a^2}$.

23. $\left(\dfrac{b}{3a - b} + \dfrac{3a}{3a - b}\right) \cdot \dfrac{3a - b}{9a^2 + b^2} \div \left(\dfrac{1}{3a - b} - \dfrac{1}{3a + b}\right)$.

24. $\dfrac{63x^2y^2}{m + n} \div \left\{\dfrac{14x(m - n)}{15(a + b)} \div \left[\dfrac{4(a - b)}{5x^2y} \div \dfrac{16(a^2 - b^2)}{7(m^2 - n^2)}\right]\right\}$.

VI. COMPLEX FRACTIONS

136. A fraction whose numerator, denominator, or both are fractional is called a **complex fraction**.

E.g., $\dfrac{\dfrac{a+b}{c}}{a^2+ab+b^2}$, $\dfrac{a}{\dfrac{b+c}{bc}}$, $\dfrac{\dfrac{a+b}{a}}{\dfrac{a-b}{a}}$ are complex fractions.

137. Complex fractions are simplified either by performing the division indicated, or by multiplying both terms by such a factor as shall render them integral.

E.g.,
$$\frac{\dfrac{a+b}{c}}{\dfrac{a^2-b^2}{c^2}}=\frac{c^2}{a^2-b^2}\cdot\frac{a+b}{c} \qquad \S\ 135$$
$$=\frac{c}{a-b}. \qquad \S\S\ 130,\ 118$$

Or,
$$\frac{\dfrac{a+b}{c}}{\dfrac{a^2-b^2}{c^2}}=\frac{c(a+b)}{a^2-b^2},\text{ by multiplying both terms by } c^2,$$
$$=\frac{c}{a-b},\text{ by cancelling } a+b.$$

It is obvious that the latter plan is the better when the multiplying factor is easily seen.

E.g., to simplify $\dfrac{\dfrac{x^2-y^2}{xy}}{\dfrac{x-y}{y^2}}$.

Multiplying both terms by xy^2, this equals

$$\frac{y(x^2-y^2)}{x(x-y)}=\frac{y(x+y)}{x}.$$

EXERCISE LVII

Simplify the following complex fractions:

1. $\dfrac{\frac{a}{bc}}{\frac{c}{ab}}$.

2. $\dfrac{\frac{a^2bc}{a+b}}{\frac{a-b}{bc}}$.

3. $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}}$.

4. $\dfrac{\frac{a}{xyz}}{\frac{a^3}{x^3y^3z^3}}$.

5. $\dfrac{\frac{x}{x-y}}{\frac{x^2}{x^2-y^2}}$.

6. $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{a^2}{b}-\frac{b^2}{a}}$.

7. $\dfrac{\frac{a+b}{a}}{\frac{a-b}{b}}$.

8. $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x^2}+\frac{1}{y^2}}$.

9. $\dfrac{\frac{m}{(m+n)^2}}{\frac{m^2}{m+n}}$.

10. $\dfrac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}$.

11. $\dfrac{\frac{x^2-y^2}{x}}{\frac{x+y}{y}}$.

12. $\dfrac{\frac{3(a+b)}{b}}{\frac{a^3}{b^3}-1}$.

13. $\dfrac{1+\frac{a}{1+a}}{a+\frac{1}{1+a}}$.

14. $\dfrac{\frac{a+1}{a-1}+\frac{a-1}{a+1}}{\frac{a+1}{a-1}-\frac{a-1}{a+1}}$.

15. $\dfrac{\frac{1}{a^4+a^2b^2+b^4}}{\frac{1}{a^3+b^3}\cdot(a+b)}$.

16. $\dfrac{\left(1+\frac{a}{b}\right)\left(2+\frac{b}{a}\right)}{1+\frac{a}{b}+\frac{b}{a}}$.

17. $\dfrac{1}{\left(1+\frac{y}{x}\right)\left(1-\frac{z}{x}\right)}$.

18. $\dfrac{a(a-b)-b(a+b)}{\frac{a}{a+b}-\frac{b}{a-b}}$.

19. $\dfrac{8}{\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\right)}$.

138. Continued fractions. Complex fractions of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cdots}}}$$

are called **continued fractions.**

Such fractions are usually simplified to the best advantage by first multiplying the terms of the last fraction of the form $\dfrac{c}{d+\frac{e}{f}}$ by the last denominator, f, and so working up.

E.g., to simplify the fraction $\cfrac{1}{a+\cfrac{1}{b+\frac{1}{c}}}$.

Multiplying the terms of $\dfrac{1}{b+\frac{1}{c}}$ by c, the original fraction reduces to $\dfrac{1}{a+\frac{c}{bc+1}}$. Multiplying the terms of this fraction by $bc+1$, this reduces to $\dfrac{bc+1}{abc+a+c}$.

Check. Let $a=b=c=1$. Then $\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$.

EXERCISE LVIII

Simplify the following:

1. $1+\dfrac{1}{1+\frac{1}{2}}$.

2. $x-\dfrac{x}{y+\frac{x}{y}}$.

3. $a+\dfrac{a}{a+\frac{a}{b}}$.

4. $\dfrac{x}{x+\frac{x}{y+x}}$.

5. $1+\dfrac{2}{3+\frac{4}{5}}$.

6. $a+\dfrac{b}{c+\dfrac{d}{e}}$.

7. $\dfrac{a}{b-\dfrac{c}{a-\dfrac{1}{b}}}$.

8. $\dfrac{m}{n-\dfrac{m}{n-\dfrac{m}{n}}}$.

9. $\dfrac{1}{a+\dfrac{a-b}{a+\dfrac{b}{a}}}$.

10. $\dfrac{\dfrac{a}{b}}{1+\dfrac{a}{1-\dfrac{b^2}{a^2}}}$.

11. $\dfrac{m}{m-\dfrac{m}{m-\dfrac{m}{2}}}$.

12. $\dfrac{a-b}{a+b+\dfrac{a}{a+\dfrac{b}{a}}}$.

13. $\dfrac{1}{x-\dfrac{x^2-1}{x+\dfrac{1}{x-1}}}$.

14. $\dfrac{1}{x-\dfrac{1+x^2}{x-\dfrac{1}{1-x}}}$.

15. $\dfrac{1}{x^2-\dfrac{x^3-1}{x+\dfrac{1}{x+1}}}$.

16. $a^2-b^2+\dfrac{1}{a^2-\dfrac{b^2}{a^2}}$.

17. $\dfrac{x+y}{x+y+\dfrac{1}{x-y+\dfrac{1}{x+y}}}$.

18. $a^3+\dfrac{a^2}{a^2+\dfrac{1}{a^3-\dfrac{a^8+a^3-1}{a^5}}}$.

19. $\dfrac{a}{1+\dfrac{a}{1+a+\dfrac{a}{1+a+a^2}}}$.

VII. FRACTIONS OF THE FORM $\frac{0}{0}$, $\frac{a}{0}$, AND $\frac{a}{\infty}$

139. By the definition of fraction (§ 115) expressions of division in which the divisor (denominator) is zero were excluded. An interpretation of this exceptional case will now be given.

When the absolute value of a variable quantity can exceed any given positive number, the quantity is said to increase *without limit*, or indefinitely.

E.g., in the series $\frac{1}{1}, \frac{1}{0.1}, \frac{1}{0.01}, \frac{1}{0.001}, \cdots$, the values of the successive terms are 1, 10, 100, $\cdots$. Hence, as the absolute values of the denominators are getting smaller, the absolute values of the fractions are getting larger and may be made to increase without limit.

140. The symbol for an infinitely great quantity is ∞, read "**infinity.**"

141. If a is a constant finite quantity, the absolute value of $\frac{a}{x}$ can be made as small as we please by increasing x sufficiently. That is, $\frac{a}{x}$ can be brought as near 0 as we please. This is expressed by saying that the *limit* of $\frac{a}{x}$, as x increases indefinitely, is 0.

142. The form $\frac{0}{0}$. The fraction $\frac{x^2-a^2}{x-a}$ has a meaning for all values of x except $x=a$. If $x=a$, however, the fraction becomes $\frac{0}{0}$, and the meaning of this symbol will now be considered.

$$\frac{x^2-a^2}{x-a}=\frac{x-a}{x-a}\cdot(x+a)=x+a;$$

and as x approaches as its limit a, this approaches $a + a$ or $2a$. Hence, we say that the *limit* of $\frac{x^2 - a^2}{x - a}$, as x approaches a, is $2a$.

In the same way the limit of

$$\frac{x^2 - 4x + 4}{x - 2} = \frac{x - 2}{x - 2} \cdot (x - 2) = x - 2,$$

as x approaches 2, is evidently 0.

But all these fractions approach the form $\frac{0}{0}$ as x approaches the limit assigned, and in the several cases the fractions approach different limits. And since the limits are undetermined at first sight, $\frac{0}{0}$ ***is said to stand for an undetermined expression.***

This is commonly expressed by saying that $\frac{0}{0}$ is *indeterminate.* The limit, however, can often be determined.

The symbol $\doteq$ is read "approaches as its limit."

143. The fact that the limit of $\frac{x^2 - 1}{x - 1}$ is 2 as $x \doteq 1$ is expressed in symbols thus:

$$\left.\frac{x^2 - 1}{x - 1}\right]_1 \doteq 2.$$

144. In general, *to find the limit which a fraction approaches as each term approaches* 0, *reduce the fraction to its lowest terms.*

Thus, $$\left.\frac{x^3 - 1}{x - 1}\right]_1 \doteq x^2 + x + 1]_1 \doteq 3.$$

145. The form $\frac{a}{0}$. This form should be interpreted to mean an expression whose absolute value is infinite.

For in the fraction $\frac{a}{x}$, as $x \doteq 0$ the absolute value of the fraction increases without limit.

146. The form $\frac{a}{\infty}$. This form should be interpreted to mean an expression whose absolute value is zero.

For as x increases without limit, $\frac{a}{x} \doteq 0$.

EXERCISE LIX

Find the limit of each of the following expressions:

1. $\left.\dfrac{x^3+1}{x+1}\right]_{-1}$.

2. $\left.\dfrac{x^5-32}{x-2}\right]_{2}$.

3. $\left.\dfrac{x^2-a^2}{x-a}\right]_{a}$.

4. $\left.\dfrac{x^2-4m^2}{x-2m}\right]_{2m}$.

5. $\left.\dfrac{4x^2-b^2}{2x-b}\right]_{\frac{b}{2}}$.

6. $\left.\dfrac{x^2-5x-14}{x-7}\right]_{7}$.

7. $\left.\dfrac{x^2-2x-3}{x-3}\right]_{3}$.

8. $\left.\dfrac{x^2-4x-5}{x+1}\right]_{-1}$.

9. $\left.\dfrac{x^2-5x-6}{x-6}\right]_{6}$.

10. $\left.\dfrac{x^2+4x-21}{x+7}\right]_{-7}$.

11. $\left.\dfrac{x^2+2x-8}{x-2}\right]_{2}$.

12. $\left.\dfrac{2x^2-5x+2}{2x-1}\right]_{\frac{1}{2}}$.

13. $\left.\dfrac{x^2+2x-8}{x+4}\right]_{-4}$.

14. $\left.\dfrac{x^2+10x+24}{x+6}\right]_{-6}$.

15. $\left.\dfrac{x^2+4x-21}{x-3}\right]_{3}$.

16. $\left.\dfrac{6x^2-13x+6}{3x^2+x-2}\right]_{\frac{2}{3}}$.

17. $\left.\dfrac{2x^2-5x+2}{x-2}\right]_{2}$.

18. $\left.\dfrac{x^3-2x^2-5x+6}{x^2-4x+3}\right]_{1}$.

19. $\left.\dfrac{x^2+8x+15}{x+5}\right]_{-5}$.

20. $\left.\dfrac{x^3+2x^2+2x+1}{x+1}\right]_{-1}$.

REVIEW EXERCISE LX

1. If $w=1$, $x=0.5$, $y=3$, $z=0.2$, find the value of $w-[2w-3x-\{4w-5x-6y-(7w-8x-9y-10z)\}]$.

2. Reduce to its lowest terms $\dfrac{5a^3+2a^2-15a-6}{7a^3-4a^2-21a+12}$.

3. From $\dfrac{23a^2+18ab+17b^2}{12a^2+31ab+20b^2}$ subtract $\dfrac{2a+3b}{3a+4b}$.

4. Find the highest common factor of $1-a+a^2-a^3+a^4-a^5$ and $1-a^3$.

5. Multiply $\dfrac{a^2+5a+6}{a^2+5a}$ by $\dfrac{a^2+6a+5}{a^2+3a}$.

6. If $2s=a+b+c$, prove that $[(s-a)+(s-b)]^3=(s-a)^3+(s-b)^3+3(s-a)(s-b)c$.

7. Add $\dfrac{a^2-a+1}{a^2+a+1}$, $\dfrac{2a(a-1)^2}{a^4+a^2+1}$, and $\dfrac{2a^2(a^2-1)^2}{a^8+a^4+1}$.

8. Find the highest common factor of $5a^3-18a^2b+11ab^2-6b^3$ and $7a^2-23ab+6b^2$.

9. Factor $49a^2+14a-3$.

10. Show that $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc=(a+b)(b+c)(c+a)$.

11. Simplify $\dfrac{\dfrac{a-b}{2b+a}+\dfrac{1}{2}}{3\frac{1}{2}-\dfrac{5a+7b}{a+2b}}$.

12. Simplify $\dfrac{\dfrac{1+a}{1+a^2}-\dfrac{1+a^2}{1+a^3}}{\dfrac{1+a^2}{1+a^3}-\dfrac{1+a^3}{1+a^4}}$.

13. Simplify $x+y+\cfrac{1}{x+y+\cfrac{1}{x+y+\cfrac{1}{x+y}}}$.

CHAPTER VIII

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY

I. INTEGRAL EQUATIONS

147. An **equation** has already been defined (§ 16) as an equality which exists only for particular values of certain letters representing the **unknown quantities.**

E.g., $x + 4 = 0$ exists only for the value $x = -4$.

148. An equation is said to be **rational, irrational, integral,** or **fractional,** according as the two members, when like terms are united, are so with respect to the unknown quantity.

Thus $x + 2 = 0$ is rational and integral;

$2 + \sqrt{x} = 0$ is irrational;

$\frac{2}{x} + 3 = 5$ is fractional.

149. A rational integral equation containing no term above the first degree is called a **simple** or **linear** equation, as in the case of $2x + 3 = 9$.

150. **Known and unknown quantities.** It is the custom to represent the *unknown quantities* in an equation by the *last letters* of the alphabet, particularly by x, y, z.

151. Quantities whose values are supposed to be *known* are generally represented by the *first letters* of the alphabet, as by a, b, c, $\cdots$.

E.g., in the equation $ax+b=0$, a and b are supposed to be known. Dividing both members by a, $x+b/a=0$, which is satisfied if $x=-b/a$.

152. The *solution* of the simple equation has already been explained (§ 17). The general case, where only integral coefficients are considered, will be understood from the following problems.

ILLUSTRATIVE PROBLEMS

1. Given the equation $5x-2=3x+8$, to find the value of x.

1.	$5x-2=3x+8$.	Given.	
2.	$5x=3x+10$.	Adding 2.	Ax. 2
3.	$2x=10$.	Subtracting $3x$.	Ax. 3
4.	$x=5$.	Dividing by 2.	Ax. 7

Check. Substitute 5 for x in the *original equation*, and

$$25-2=15+8,$$

or

$$23=23.$$

2. Given the equation $2ax-a^2=ax+3a^2$, to find the value of x.

1.	$2ax-a^2=ax+3a^2$.	Given.	
2.	$2ax=ax+4a^2$.	Adding a^2	Ax. 2
3.	$ax=4a^2$.	Subtracting ax.	Ax. 3
4.	$x=4a$.	Dividing by a.	Ax. 7

Check. Substitute $4a$ for x in the original equation, and

$$8a^2-a^2=4a^2+3a^2.$$

153. From these illustrative problems it will be observed that any term may be transferred from one member of an equation to the other if its sign is changed. This operation is called **transposition.**

E.g., if $x + 2 = 7$, transposing 2 we have

$$x = 7 - 2,$$

or

$$x = 5.$$

154. General directions for solution. From the suggestions given, it appears that, to solve a simple integral equation, we

1. *Transpose the terms containing the unknown quantities to the first member, changing the signs* (Axs. 2, 3);
2. *Transpose the terms containing only known quantities to the second member, changing the signs;*
3. *Unite terms;*
4. *Divide by the coefficient of the unknown quantity.*

The result may be checked by substituting in the original equation.

EXERCISE LXI

Solve the following equations:

1. $ax + b = bx + a.$
2. $4x + 2 = 2x + 10.$
3. $17 - x = 2x - 1.$
4. $(2 - x)(5 - x) = x^2.$
5. $5x + 7 = x + 27.$
6. $4x - 34 = 22 - 3x.$
7. $2x + 3 = 4x + 5.$
8. $3x - 2(2 - x) = 21.$
9. $3x - 5 = 5x - 11.$
10. $12x - 28 = 8 + 3x.$
11. $8 - 3x = 7x - 72.$
12. $3(x - 1) = 4(x + 1).$
13. $8x - (7 - x) = 29.$
14. $27x - 127 = 11 - 19x.$

15. $2x + 3 - (x - 3) = 5.$

16. $x + 2 - 3(x - 4) = 0.$

17. $8x - (7 + x) = 7 + 2x.$

18. $9x - 3(5x - 6) = -30.$

19. $2(x + 3) - 3(x + 2) = 0.$

20. $x(x^2 + 1) = x(x^2 - 1) + 9.$

21. $(x + 5)^2 = 21x + (4 - x)^2.$

22. $x + 2 + 3x + 4 = 5x + 6.$

23. $3x + 4x + 5x = 6x + 72.$

24. $x + 1 - 2(x - 1) = x - 10.$

25. $1 + 2x - 3 + 4x = 5 - 6x.$

26. $10x + 4 - 8x + 3 = x + 17.$

27. $3x - 4 - (x - 10) = x + 10.$

28. $3x + 14 - 5(x - 3) = 4(x + 3).$

29. $x(x - 1) - x(x - 2) = 2(x - 3).$

30. $2(x - 2\overline{x - 2}) = 3(x - 3\overline{x - 3}).$

31. $2(x - 1) + 3(x - 2) = 4(x - 5).$

32. $x(1 + x + x^2) = x^3 + x^2 + 3x - 17.5.$

33. $(x + 1)(x + 2) = (x + 3)(x + 4) - 50.$

34. $2(x + 1) + 3(x + 2) + 4(x + 3) = 101.$

35. $5(x - 2) + 6(x - 1) - 7(x - 5) = 55.$

36. $2(x - 3) + 3(x - 2) - 4(x - 5) = 15.$

37. $4(x + 9) - 5(x + 10) + 6(x + 8) = 0.$

38. $3(x - 1) + 4(x + 1) - 5(2x - 7) = 21.$

II. FRACTIONAL EQUATIONS

155. If the equation contains fractions, these may be removed by multiplying both members by the lowest common multiple of the denominators. This is called **clearing the equation of fractions.**

It is not always advisable, however, to clear the equation of fractions at once, as is seen in the following illustrative problems.

ILLUSTRATIVE PROBLEMS

1. An equation which should be cleared of fractions at once: $\frac{x-3}{15}+\frac{x+7}{4}=12.$

1. Multiplying by 15·4, by ax. 6,

$$4x-12+15x+105=720.$$

2. Subtracting (transposing) $-12+105$, and uniting,

$$19x=627.$$

3. Dividing by 19, $x=33.$

Check.
$$\frac{33-3}{15}+\frac{33+7}{4}=\frac{30}{15}+\frac{40}{4}$$
$$=2+10$$
$$=12.$$

2. An equation which need not be cleared of fractions at once: $x-\frac{x}{3}-\frac{3}{4}=\frac{23}{12}.$

1. Adding $\frac{3}{4}$ and uniting terms,

$$\tfrac{2}{3}x=\tfrac{8}{3}.$$

2. Multiplying both members by $\frac{3}{2}$ (or dividing both members by $\frac{2}{3}$),

$$x=4.$$

Check. $4-\frac{4}{3}-\frac{3}{4}=\frac{23}{12}.$

3. An equation which should be cleared of fractions part at a time: $\frac{3x+7}{15}-\frac{2x-4}{7x-12}=\frac{x+1}{5}$.

1. Multiplying both members by 15,

$$3x+7-\frac{15(2x-4)}{7x-12}=3x+3. \qquad \text{Ax. 6}$$

2. Transposing and uniting terms,

$$4=\frac{15(2x-4)}{7x-12}. \qquad \text{Axs. 2, 3}$$

3. Dividing by 2 and multiplying by $7x-12$,

$$14x-24=15x-30. \qquad \text{Axs. 6, 7}$$

4. Adding $24-15x$,

$$-x=-6. \qquad \text{Ax. 2}$$

5. Multiplying by -1,

$$x=6. \qquad \text{Ax. 6}$$

Check. $\frac{25}{15}-\frac{8}{30}=\frac{7}{5}$.

4. An equation in which the fractions may be united to advantage before clearing: $\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}$.

1. Adding the fractions in each member separately,

$$\frac{x^2-x-x^2+x+2}{(x-2)(x-1)}=\frac{x^2-15x+56-x^2+15x-54}{(x-6)(x-7)}.$$

2. $$\therefore \frac{2}{(x-2)(x-1)}=\frac{2}{(x-6)(x-7)}.$$

3. Dividing by 2 and clearing of fractions,

$$x^2-13x+42=x^2-3x+2.$$

4. $$\therefore -10x=-40. \qquad \text{(Why?)}$$

5. $$\therefore x=4. \qquad \text{(Why?)}$$

Check. $\frac{4}{2}-\frac{5}{3}=\frac{-4}{-2}-\frac{-5}{-3}$.

156. General directions for solution.

1. *See if you can simplify before clearing of fractions.*

2. *Keep the denominators in factored form as long as possible.*

3. *After clearing of fractions, solve as in integral equations* (§ 154).

EXERCISE LXII

Solve the following equations:

1. $\dfrac{ab}{ax+bx}=1.$

2. $\dfrac{x-1}{x+1}=\dfrac{x-6}{x-3}.$

3. $\dfrac{50}{4x}+\dfrac{12}{x}=\dfrac{49}{10}.$

4. $\dfrac{x}{a}=x-a+\dfrac{1}{a}.$

5. $\dfrac{x}{2}+\dfrac{x}{3}=13-\dfrac{x}{4}.$

6. $\dfrac{x}{5}+\dfrac{x}{8}=17-\dfrac{x}{10}.$

7. $\dfrac{2x^2}{x-3}=2x+15.$

8. $\dfrac{4x}{3}-\dfrac{5x}{7}=x-8.$

9. $\dfrac{x-a}{x-b}=\dfrac{(2x-a)^2}{(2x-b)^2}.$

10. $\dfrac{1+\dfrac{4bx}{a}}{bx}=\dfrac{\dfrac{4a}{bx}+1}{a}.$

11. $0.5x+0.25x=1.5.$

12. $\dfrac{x-a}{x+a}+\dfrac{3b-x}{2b+x}=0.$

13. $\dfrac{6x+7}{12}=\dfrac{3x-4}{4x-3}+\dfrac{x}{2}.$

14. $\dfrac{\dfrac{x}{5}+\dfrac{1}{2}}{3}-\dfrac{x-\dfrac{x}{2}}{2}+\dfrac{x}{5}=0.$

15. $1-\dfrac{a}{a+x+\dfrac{x^2}{a-x}}=\dfrac{b}{a}.$

16. $\dfrac{x-2}{3}-\dfrac{x-4}{5}=\dfrac{x-6}{7}.$

17. $\dfrac{1}{x-3}+\dfrac{3}{x-9}=\dfrac{4}{x-6}.$

18. $\dfrac{b+x}{b+a}+\dfrac{b-x}{b-a}=\dfrac{b^2-ax}{a^2-b^2}.$

19. $\frac{5x+10.5}{x+0.5}+\frac{2x}{2x+1}=9.$

20. $\frac{\frac{2}{3}}{\frac{2}{3}+x}-\frac{2}{3}=\frac{2}{3}-\frac{\frac{2}{3}x+\frac{2}{3}}{\frac{2}{3}+x}.$

21. $\frac{x}{2}-\frac{5x+4}{3}=7-\frac{8x-2}{3}.$

22. $\frac{8}{x+3}-\frac{9}{2x+6}=\frac{15}{7x+2}.$

23. $\frac{7x+5}{6}-\frac{5x-6}{4}=\frac{8-5x}{12}.$

24. $1=\frac{a}{b}\left(1-\frac{a}{x}\right)+\frac{b}{a}\left(1-\frac{b}{x}\right).$

25. $\frac{2x-5}{6}+\frac{6x+3}{4}=5x-\frac{35}{2}.$

26. $\frac{2x+3}{5}-\frac{6x+22}{15}=\frac{3x+17}{5(1-x)}.$

27. $\frac{1}{a-b}+\frac{a-b}{x}=\frac{1}{a+b}+\frac{a+b}{x}.$

28. $\frac{1}{x-1}-\frac{2}{x-2}=\frac{3}{x-3}-\frac{4}{x-4}.$

29. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}.$

30. $\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5.$

31. $\frac{1}{(x+3)(x+5)}=\frac{1}{(x+9)(x-5)}.$

32. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=\frac{14(2x-3)^2}{25-49x^2}.$

33. $\frac{2}{2x-1}-\frac{1}{x-3}-\frac{1}{x}+\frac{2}{2x-5}=0.$

34. $\dfrac{x-1}{7}+\dfrac{x-2}{9}=\dfrac{x}{10}+\dfrac{x+4}{8}.$

35. $\dfrac{x-6}{3}+\dfrac{x+6}{6}=\dfrac{x}{4}+\dfrac{x+4}{8}.$

36. $\dfrac{x-1}{5}+\dfrac{x+1}{6}=\dfrac{x-3}{4}+\dfrac{x+3}{7}.$

37. $\dfrac{x-4}{3}+\dfrac{x+4}{7}=\dfrac{x-5}{5}+\dfrac{x+20}{10}.$

38. $\dfrac{x+5}{2}+\dfrac{x-5}{5}=\dfrac{20-x}{5}+\dfrac{7+x}{2}.$

39. $\dfrac{x+2}{8}+\dfrac{x+4}{9}=\dfrac{x+6}{10}+\dfrac{x+10}{12}.$

40. $\frac{1}{3}\{\frac{1}{3}[\frac{1}{3}(\frac{1}{3}\overline{\tfrac{1}{3}x-1})-1]-1\}-1=0.$

41. $5x-\dfrac{2x-1}{3}+1=3x+\dfrac{x+2}{2}+7.$

42. $\dfrac{5}{5x+8}-\dfrac{4}{2x+3}=\dfrac{1}{5}\left(\dfrac{3}{x+3}-\dfrac{8}{x+2}\right).$

43. $1-\dfrac{x}{2}\left(1-\dfrac{3}{4x}\right)=\dfrac{5x}{6}\left(7-\dfrac{6}{7x}\right)-35\frac{51}{56}.$

44. $\dfrac{3x^2-2x+1}{5}=\dfrac{(7x-2)(3x-6)}{35}+\dfrac{9}{10}.$

45. $\dfrac{a(a+b)x}{a^2-b^2}-\dfrac{a^2-b^2}{a+b}-\dfrac{2bx}{b-a}=\dfrac{(5a+b)b}{a-b}.$

46. $\frac{1}{2}x-\frac{3}{4}+\frac{5}{6}x-\frac{7}{8}=\frac{9}{10}+\frac{11}{12}x-\frac{13}{14}-\frac{15}{16}x.$

47. $\dfrac{13x-10}{36}+\dfrac{4x+9}{18}-\dfrac{7(x-2)}{12}=\dfrac{13x-28}{17x-66}.$

48. $\dfrac{a}{x+1}-\dfrac{2(3a+5)}{x-1}+\dfrac{8a+15}{x-2}=\dfrac{3(a+2)}{x-3}-\dfrac{1}{x+2}.$

49. $\frac{1}{15}(2x-1)-\frac{1}{16}(3x-2)=\frac{1}{18}(x-12)-\frac{1}{24}(x+1).$

III. APPLICATION OF SIMPLE EQUATIONS

ILLUSTRATIVE PROBLEMS

1. The sum of two numbers is 200, and their difference is 50. Find the numbers.

1. Let	x	= the lesser number.
2. Then	$x + 50$	= the greater number.
3. And	$x + x + 50$	= the sum.
4. But	200	= the sum.
5.	$\therefore\ x + x + 50 = 200.$	
6.	$\therefore\ x = 75,$	
and	$x + 50 = 125.$	

Check. The sum of 125 and 75 is 200, and their difference is 50.

In checking, always substitute in the *problem* instead of the equation, because there may have been an error in forming the equation. The neglect to take this precaution often leads to wrong results.

2. What number must be added to the two terms of the fraction $\frac{7}{23}$ in order that the resulting fraction shall equal $\frac{59}{67}$?

1. Let x = the number to be added.

2. Then $\dfrac{7 + x}{23 + x} = \dfrac{59}{67}.$

3. $\therefore\ 67(7 + x) = 59(23 + x)$, by multiplying by $(23 + x) \cdot 67$.

4. $\therefore\ 469 + 67\,x = 1357 + 59\,x.$

5. $\therefore\ 8\,x = 888$, by subtracting $59\,x$ and 469.

6. $\therefore\ x = 111.$

Check. $\dfrac{7 + 111}{23 + 111} = \dfrac{118}{134} = \dfrac{59}{67}.$ That is, if 111 is added to both terms of the fraction $\frac{7}{23}$, the result equals $\frac{59}{67}$.

3. What sum gaining $6\frac{1}{4}\%$ of itself in a year amounts to \$157.50 in 2 yrs.?

1. Let $x =$ the *number* of dollars.
2. Then $6\frac{1}{4}\% x =$ the *number* of dollars of interest for 1 yr.
3. $\therefore x + 2 \cdot 6\frac{1}{4}\% x = 157.50$ (Why?)
4. $\therefore 1.12\frac{1}{2} x = 157.50$
5. $\therefore x = 140.$ (Why?)

Check. The interest on \$140 for 2 yrs. at $6\frac{1}{4}\%$ is \$17.50, and hence the amount is \$157.50.

4. The cost of an article is \$17.15, and this is 30% less than the marked price. What is the marked price?

1. Let $x =$ the *number* of dollars of marked price.
2. Then $30\% x =$ the *number* of dollars of discount.
3. $\therefore x - 30\% x = 17.15.$
4. $\therefore 0.7 x = 17.15.$ (Why?)
5. $\therefore x = 24.50.$ (Why?)

Check. \$24.50 less 30% of \$24.50 is \$17.15.

EXERCISE LXIII

1. What number is that which when divided by 12 gives the same result as when added to 12?

2. What number is that which when multiplied by 16 gives the same result as when added to 16?

3. What number is that which when multiplied by n gives the same result as when added to n?

4. What number is that which when subtracted from 28 gives the same result as when divided by 28?

5. What number is that which when subtracted from n gives the same result as when divided by n?

6. What number is that which when subtracted from 25 gives the same result as when multiplied by 25?

7. Six is $\frac{2}{3}$ less than 10 times a certain number. Required the number.

8. Divide the number 121 into two parts such that the greater exceeds the less by 73.

9. Divide the number n into two parts such that the greater exceeds the less by a.

10. What number must be added to 3 and 7 so that the first sum shall be $\frac{3}{4}$ of the second?

11. What number must be added to a and b so that the first sum shall be $\frac{m}{n}$ of the second?

12. If from 3 times a certain number 1 be subtracted, the remainder is 11. Required the number.

13. What is that number which when multiplied by 2 and added to 15 equals the sum of the number and 20?

14. If to two-thirds of a certain number 8 be added, the sum is 10 more than half the number. Required the number.

15. If from three-fourths of a certain number 3 be subtracted, the remainder is 4 less than the number. Required the number.

16. What is that number from 4 times which if 5 be subtracted, the difference is 20 more than the remainder after taking 4 from the number?

17. What is that number from 3 times which if 8 be subtracted the result is twice the remainder after taking 1 from the number?

18. The sum of a certain number, its double, and its triple equals the remainder after subtracting the number from 7. Required the number.

19. What is the sum which diminished by $9\frac{1}{2}\%$ of itself equals \$1538.50?

20. How long will it take an investment of \$6024 to amount to \$7658.01 at $3\frac{1}{2}\%$ simple interest?

21. Divide the number n into three parts such that the first exceeds the second by p and the third by q.

22. The square of a certain number is 1188 larger than that of 6 less than the number. What is the number?

23. Divide the number 121 into three parts such that the first exceeds the second by 85 and the second is four times the third.

24. A man invests $\frac{5}{8}$ of his capital at 4% and the rest at $3\frac{1}{2}\%$, and thus receives an annual income of \$76. What is his capital?

25. A man invests one-fourth of his capital at 5%, one-fifth at 4%, and the rest at 3%, and thus secures an annual income of \$3700. What is his capital?

26. A train travelling 30 mi. per hour takes $2\frac{2}{3}$ hrs. longer to go from Detroit to Chicago than one which goes $\frac{1}{3}$ faster. What is the distance from Detroit to Chicago?

27. The square of 13 times a certain number, less the square of 3 more than 12 times the number, equals the square of 9 less than 5 times the number. What is the number?

28. If each of the two indicated factors of the two unequal products $52 \cdot 45$ and $66 \cdot 37$ is diminished by a certain number, the two products are equal. What is the number?

29. Divide the number 99 into four parts such that if 2 is added to the first, subtracted from the second, and multiplied by the third, and if the fourth is divided by 2, the results shall all be equal.

30. A train runs 75 mi. in a certain time. If it were to run $2\frac{1}{2}$ mi. an hour faster, it would run 5 mi. farther in the same time. What is the rate of the train?

31. A loaned to B a certain sum at $4\frac{1}{2}\%$, and to C a sum \$200 greater at 5%; from the two together he received \$276 per annum interest. How much did he lend each?

32. The interest for 8 yrs. upon a certain principal is \$1914, the rate being $3\frac{1}{4}\%$ for the first year, $3\frac{1}{2}\%$ for the second, $3\frac{3}{4}\%$ for the third, and so on, increasing $\frac{1}{4}\%$ each year. What is the principal?

33. A bicyclist travelling a mi. per hour is followed, after a start of m mi., by a second bicyclist travelling b mi. per hour, $b > a$. At these rates, in how many hours after the second starts will he overtake the first?

34. A capitalist has $\frac{2}{5}$ of his money invested in mining stocks which pay him 13%, $\frac{1}{3}$ in manufacturing which pays him 9%, and the balance in city bonds which pay him 3%. What is his capital, if his total income is \$26,640?

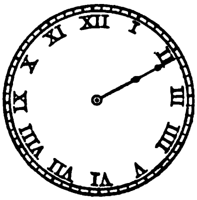
35. A man spends $\frac{1}{a}$th of his income for food, $\frac{1}{b}$th for rent, $\frac{1}{c}$th for clothing, $\frac{1}{d}$th for furniture, and saves e dollars. How much is his income?

36. Two trains start at the same time from Syracuse, one going east at the rate of 35 mi. per hour and the other going west at a rate $\frac{1}{7}$ greater. How long after starting will they, at these rates, be exactly 100 mi. apart?

37. Two trains start at the same time from Buffalo and New York, respectively, 450 mi. apart; the one from New York travels at the rate of 50 mi. per hour, and the other 0.8 as fast. How far from New York do they meet?

ILLUSTRATIVE PROBLEMS

(*a*) At what time between 2 and 3 o'clock will the hour and minute hands of a clock be together.



1. They are evidently together at 12. Let x = the number of hours after 12 before they are together the second time (*i.e.*, between 2 and 3).

2. Then after 1 hr. (at 1 o'clock) the minute hand has *gained* 11 5-min. spaces.

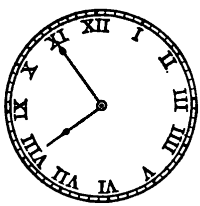
3. In x hrs. it will *gain* $11x$ 5-min. spaces.

But it has 2 revolutions to gain, or 24 5-min. spaces.

4. $$\therefore\ 11x = 24,$$
$$x = 2\tfrac{2}{11},$$

and in $2\frac{2}{11}$ hrs., or at 2 o'cl. 10 min. $54\frac{6}{11}$ sec. they will be together.

(*b*) At what time between 7 and 8 o'clock will they be at right angles to each other?



1. Let x = the number of hours after 12.

2. In order to be together the minute hand must *gain* 7 revolutions or 84 5-min. spaces.

Hence to be at right angles it must gain $84+3$ or $84-3$ 5-min. spaces.

3. $\therefore$ as in Ex. (*a*), $11x = 87$ or 81,
$$x = 7\tfrac{10}{11} \text{ or } 7\tfrac{4}{11},$$

and the time is 7 o'cl. 54 min. $32\frac{8}{11}$ sec.

or 7 o'cl. 21 min. $49\frac{1}{11}$ sec.

The same reasoning holds for cases of the hands being a 5-min. space, or any other distance apart, for the movements of the planets about the sun, and for similar problems.

38. At what time between 1 and 2 o'clock are the hands of a clock

(*a*) Together? (*b*) At right angles?

(*c*) In the same straight line in opposite directions?

39. At what time are the hands of a clock together between

(*a*) 3 and 4? (*b*) 4 and 5?

(*c*) 5 and 6? (*d*) 11 and 12?

40. At what time between 5 and 6 o'clock are the hands of a clock

(*a*) At right angles? (*b*) A 5-min. space apart?

41. At what time or times are the hands of a clock at right angles to each other between

(*a*) 6 and 7? (*b*) 8 and 9?

(*c*) 2 and 3? (*d*) 11 and 12?

42. At what time are the hands of a clock in the same straight line in opposite directions

(*a*) Between 4 and 5? (*b*) Between 9 and 10?

(*c*) Between 10 and 11? (*d*) Between 11 and 12?

43. At what time are the hands of a clock a 1-min. space apart for the first time after 12?

44. The planet Venus passes about the sun 13 times to the earth's 8. How many months from the time when Venus is between the earth and the sun to the next time when it is in the same relative position?

45. Two bodies start at the same time from two points 243 in. apart, and move towards each other, one at the rate of 5 in. per second, and the other 2 in. per second faster. In how many seconds will they be 39 in. apart?

46. Seen from the earth, the moon completes the circuit of the heavens in 27 das. 7 hrs. 43 mins. 4.68 secs., and the sun in 365 das. 5 hrs. 48 mins. 47.8 secs., in the same direction. Required the time to 0.0001 da. from one full moon to the next, the motions being supposed to be uniform.

Many problems which were of considerable difficulty prior to the introduction of our present algebraic symbols, about the opening of the seventeenth century, are now comparatively easy. They have considerable historical interest as showing the state of the science at various periods, and a few examples are here inserted.

ILLUSTRATIVE PROBLEM

A cistern is supplied by 2 pipes; the first can fill it in 4 hrs., and the second in 3 hrs. How long will it take both together to fill it?

1. Let $x =$ the number of hours.
2. $\because$ the first requires 4 hrs., it will fill $\frac{1}{4}$ in 1 hr.
   $\because$ the second requires 3 hrs., it will fill $\frac{1}{3}$ in 1 hr.
3. In x hrs. the two will fill the whole,

$$\therefore x(\tfrac{1}{4} + \tfrac{1}{3}) = 1,$$

$$\therefore \tfrac{7}{12}x = 1,$$

4. $$\therefore x = 1\tfrac{5}{7}.$$
5. Hence it will take $1\frac{5}{7}$ hrs. for the two to fill it.

47. Demochares lived $\frac{1}{4}$ of his life as a boy, $\frac{1}{5}$ as a young man, $\frac{1}{3}$ as a man, and 13 years as an old man. How old was he then? (Metrodorus, 325 A.D.)

48. A steamer can run 25 mi. an hour in still water. If it can run 90 mi. with the current in the same time that it can run 60 mi. against it, what is the rate of the current? (As applied to row-boats, a very old problem.)

49. Of 4 pipes, the first fills a cistern in 1 da., the second in 2 das., the third in 3 das., and the fourth in 4 das.? How long will it take all running together to fill it? (Heron of Alexandria, about the beginning of the Christian Era.)

50. A cistern is supplied by 3 pipes; the first can fill it in 5 hr., the second in 6 hr., and the third in 7 hr. How long will it take all three together to fill it?

51. In the centre of a pond 10 ft. square grew a reed 1 ft. above the surface; but when the top was pulled to the bank it just reached the edge of the surface. How deep was the water? (From an old Chinese arithmetic, Kiu chang, about 2600 B.C.)

52. Heap, its whole, its seventh, it makes 19. (That is, what is the number which when increased by its seventh equals 19? From the mathematical work copied by the Egyptian Ahmes about 1700 B.C. from a papyrus written about a thousand years earlier.)

53. Find the number, $\frac{1}{3}$ of which and 1, multiplied by $\frac{1}{4}$ of which and 2, equals the number plus 13. (Mohammed ben Musa Al-Khowarazmi, the famous Persian mathematician, 830 A.D. From the title of his book comes the word Algebra, and from the latter part of his name — referring to his birthplace — comes our word Algorism.)

54. In a pond the top of a lotus bud reached $\frac{1}{2}$ ft. above the surface, but blown by the wind it just reached the surface at a point 2 ft. from its upright position. How deep was the water? (From a mathematical work by Bhaskara, a Hindu writer of about 1150 A.D. The work was named the Lilavati in honor of his daughter.)

55. A horse and a donkey, laden with corn, were walking together. The horse said to the donkey: "If you gave me one measure of corn, I should carry twice as much as you; but if I gave you one, we should carry equal burdens." Tell me their burdens, O most learned master of geometry. (Attributed to Euclid, the great writer on geometry at Alexandria, about 300 B.C.)

REVIEW EXERCISE LXIV

1. Divide $x^2-3\,ax-2\,a^2+\dfrac{12\,a^3}{x+3\,a}$ by $3\,x-6\,a-\dfrac{2\,x^2}{x+3\,a}$.

2. Solve the equation $\frac{1}{15}(1+4\,x)-\frac{1}{3}(5\,x-1)=x-2$.

3. Solve the equation

$$(x+2.5)(x-1.5)-(x+5)(x-3)+0.75=0.$$

4. Simplify the expression

$$a-\{2\,b+[3\,c-3\,a-(a+b)]\}+2\,a-(b+3\,c).$$

5. Find the highest common factor of

$$a^5+3\,a^4-8\,a^2-9\,a-3 \text{ and } a^5-2\,a^4-6\,a^3+2\,a^2+13\,a+6.$$

6. Reduce to lowest terms the fraction

$$\frac{ab+2}{2\,a+(a^2-4)\,b-2\,ab^2}.$$

7. The difference of the squares of two consecutive numbers is 19. Required the numbers.

8. A cistern is filled by 3 pipes; by the first alone it can be filled in 80 mins., by the second alone in 200 mins., and by the third alone in 5 hrs. In what time can it be filled if all are open together?

9. A person bought a picture and a frame, the frame costing the same as the picture. If the frame had cost $1 less and the picture 75 cts. more, the price of the frame would have been only half that of the picture. Find the cost of each.

10. The cost of publication of each copy of a certain illustrated magazine is $6\frac{1}{2}$ cts.; it sells to dealers for 6 cts., and the amount received for advertisements is 10% of the amount received for all magazines issued beyond 10,000. Find the least number of magazines which can be issued without loss.

CHAPTER IX

SIMPLE EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES

157. A *single* linear equation containing *two* unknown quantities does not furnish determinate values of these quantities.

This means a single equation in which the similar terms have been united. *I.e.*, $x + y = x + 3$ is not included, because the x's have not been united.

E.g., $x - y = 1$ is satisfied if $x = 1$ and $y = 0$, or if $x = 2$ and $y = 1$, or if $x = 3$ and $y = 2$, etc.

158. But *two* linear equations containing *two* unknown quantities furnish, in general, determinate values.

Similarly, as will be seen, a system of *three* linear equations containing *three* unknown quantities, furnishes, in general, determinate values of all three quantities.

159. Equations all of which can be satisfied by the same values of the unknown quantities are said to be **simultaneous**.

E.g., $x + y = 7$, $x - y = 3$, are two equations which are satisfied if $x = 5$ and $y = 2$. Hence they are simultaneous.

But $x + y = 7$ and $x + y = 8$ cannot be satisfied by the same values of x and y, and hence they are not simultaneous.

The equations $x + 2y = 3$, $3x + 6y = 9$, are simultaneous; but each being derivable from the other, they do not furnish determinate values.

I. ELIMINATION BY ADDITION OR SUBTRACTION

160. The solution of two simultaneous equations involving two unknown quantities is made to depend upon the solution of a single equation involving but one of the unknown quantities. The usual process, by addition or subtraction, is seen in the following solutions:

1. Solve the system of equations

$$1.\quad 4x + 3y = 41.$$

$$2.\quad 3x - 2y = 1.$$

We first seek to give the y's coefficients having the same absolute values. This can be done by multiplying both members of the first by 2, and of the second by 3. Then

3. $8x + 6y = 82.$

4. $9x - 6y = 3.$

Add equations 3 and 4, member by member, and

5. $17x = 85.$

6. $\therefore\ x = 5.$

Substitute this value in equation 1, and

7. $4 \cdot 5 + 3y = 41.$

8. $\therefore\ 3y = 21.$

9. $\therefore\ y = 7.$

Check. Substitute these values in equation 2 (because y was obtained by substituting in equation 1), and $3 \cdot 5 - 2 \cdot 7 = 1$.

161. For brevity we shall hereafter use the expressions, in solutions, "Multiply (2) by 5," etc., meaning thereby, "Multiply *both members* of equation 2 by 5," etc., and "Add the equations," meaning thereby, "Add the equations, member by member."

162. When one of the unknown quantities has been made to disappear (as in passing from steps 3 and 4 to step 5 on p. 154) it is said to be *eliminated*.

In the solution on p. 154, y was *eliminated by addition*. The quantity x may, however, be eliminated first, by subtraction, as in the following solution:

2. Solve the system of equations

1. $4x + 3y = 41.$
2. $3x - 2y = 1.$

3. $\because\ 12x + 9y = 123$, multiplying (1) by 3,
4. and $12x - 8y = 4$, multiplying (2) by 4.
5. $\therefore\ 17y = 119$, subtracting (4) from (3).
6. $\therefore\ y = 7.$
7. $\therefore\ 4x + 21 = 41$, substituting 7 for y.
8. $\therefore\ 4x = 20.$
9. $\therefore\ x = 5.$

Check. In which equation should these values now be substituted? (Why?)

Where the coefficients of x and y are positive, it is usually easier to eliminate by subtraction, as in the following solution:

3. Solve the system of equations

1. $x + 2y = 11.$
2. $2x + y = 13.$
3. $\therefore\ 2x + 4y = 22$, multiplying (1) by 2,
4. $\therefore\ 3y = 9$, subtracting (2) from (3).
5. $\therefore\ y = 3$, dividing by 3.
6. $\therefore\ 4x + 2y = 26$, multiplying (2) by 2,
7. $\therefore\ 3x = 15$, subtracting (1) from (6).
8. $\therefore\ x = 5$, dividing by 3.

Of course the value of x might have been found, as before, by substituting.

Other types are illustrated in the two problems following.

4. Solve the system of equations

$$1.\quad \frac{x}{3}-\frac{y}{2}=2.$$

$$2.\quad \frac{x}{2}+\frac{y}{4}=7.$$

It is not worth while here to clear of fractions. Simply multiply both members of the first by $\frac{1}{2}$, and

3. $\frac{x}{6}-\frac{y}{4}=1.$

4. $\therefore \frac{2x}{3}=8$, adding (2) and (3).

5. $\therefore x=12.$

It is now apparent that y can easily be found and the results checked in the usual way.

I.e., $y=4,$

and $\frac{12}{3}-\frac{4}{2}=2$, etc.

5. Solve the system of equations

$$1.\quad \frac{3}{x}+\frac{2}{y}=\frac{7}{4}.$$

$$2.\quad \frac{2}{x}+\frac{1}{y}=1.$$

Equations of this kind should not be cleared of fractions.

3. $\frac{4}{x}+\frac{2}{y}=2$, multiplying (2) by 2.

4. $\therefore \frac{1}{x}=\frac{1}{4}$, subtracting (1) from (3).

5. $\therefore 4=x$, multiplying by $4x$.

Hence, y is easily found to be 2, and the results check.

EXERCISE LXV

Solve the following systems of equations:

1. $7x - 3y = 3.$
$5x + 7y = 25.$

2. $3x + 5y = 5.$
$4x - 3y = 26.$

3. $x + 17y = 53.$
$8x + y = 19.$

4. $5x + 2y = 1.$
$13x + 8y = 11.$

5. $6x - 5y = 12.$
$12x - 11y = 27.$

6. $1.7x + 1.1y = 13.$
$1.3x - 0.1y = 1.$

7. $\frac{x}{3} - \frac{y}{7} = \frac{7}{2}.$
$\frac{x}{2} + \frac{y}{5} = 11.$

8. $\frac{x}{5} + \frac{y}{10} = 3.$
$\frac{x}{10} + \frac{y}{5} = 3.$

9. $\frac{3}{x} - \frac{4}{y} = -\frac{15}{2}.$
$\frac{2}{x} + \frac{5}{y} = \frac{31}{3}.$

10. $\frac{2x}{3} - \frac{y}{2} = 5.$
$\frac{x}{2} + \frac{2y}{3} = 20.$

11. $7x + y = 42.$
$3x - y = 8.$

12. $5x + 2y = 1.$
$8x - y = 10.$

13. $3x - 3y = 45.$
$2x + y = 18.$

14. $5x - 2y = 35.$
$4x + y = 25.$

15. $4x + y = 26.$
$3x - 2y = 25.$

16. $3x - 2y = -4.$
$2x - y = -1.$

17. $2x - 10y = 14.$
$3x - 5y = 1.$

18. $15x + 2y = 16.$
$7x - 3y = 35.$

19. $2x + 7y = -25.$
$3x - y = 20.$

20. $11x - 4y = 0.$
$4x - 11y = -105.$

II. ELIMINATION BY SUBSTITUTION AND COMPARISON

163. After finding the value of one unknown quantity by addition or subtraction the other is usually, but not necessarily, found by substitution. It is often more convenient to find each by substitution, especially when one of the coefficients is 1.

This method of **elimination by substitution** is illustrated in the following solution:

1. Given $$x - \tfrac{2}{3} y = -5,$$
2. and $$3x + 2y = 45.$$
3. From equation 1 we have: $$x = \tfrac{2}{3} y - 5.$$

Substitute this value in equation 2, and

4. $$2y - 15 + 2y = 45, \text{ from which}$$
5. $$4y = 60.$$
6. $$\therefore\ y = 15.$$

From this x is found, by substitution, to be 5.

It is not necessary that the coefficient of x or y should be 1, although this is the case in which the method is most frequently employed. Consider, for example, the following solution:

1. Given $$2x + 5y = 154,$$
2. and $$30x - 2y = 0.$$
3. From equation 2, $$x = \tfrac{1}{15} y.$$
4. Substituting, $$\tfrac{2}{15} y + 5y = 154,$$

whence $$y = 30.$$
$$\therefore\ x = 2.$$

164. A special form of substitution occurs when the value of one of the unknown quantities is found in each equation, and these values are compared. This is called **elimination by comparison.**

The method is illustrated in the following solution:

1. Given $x - \frac{2}{3}y = -5,$
2. and $3x + 2y = 45.$

Solving equations 1 and 2 for x, we have:

3. $x = \frac{2}{3}y - 5,$
4. and $x = 15 - \frac{2}{3}y.$

Substituting the value of x from step 3 in step 4, or, what is the same thing, comparing the values of x (by Ax. 1), we have:

5. $\frac{2}{3}y - 5 = 15 - \frac{2}{3}y.$
6. $\therefore \frac{4}{3}y = 20.$
7. $\therefore y = 15.$
8. $\therefore x = 5,$ by substituting in step 3.

Check. Substituting in both of the original equations,

$$5 - \frac{2}{3} \cdot 15 = -5.$$

$$3 \cdot 5 + 2 \cdot 15 = 45.$$

This method is especially valuable when the coefficients of x (or of y) are both 1, as in the following set of equations:

1. Given $x + 5y = 20,$
2. and $x - 7y = -16.$

Solving both equations for x,

3. $x = 20 - 5y,$
4. and $x = 7y - 16.$
5. $\therefore 7y - 16 = 20 - 5y.$
6. $\therefore 12y = 36.$
7. $\therefore y = 3.$

By substituting in equation 1,

8. $x = 5.$

EXERCISE LXVI

Solve the following systems of equations by substitution or comparison:

1. $x + y = s.$
 $x - y = d.$

2. $x + 5y = 36.$
 $2x - y = 6.$

3. $x + ay = b.$
 $cx + y = d.$

4. $x + 4y = 1.$
 $x - 5y = 19.$

5. $y - x = 3.$
 $2x + y = 5.$

6. $x + y = 11.$
 $x + 12y = 0.$

7. $3x + y = 30.$
 $3y + x = 34.$

8. $x + 2y = 30.$
 $\frac{1}{2}x - \frac{1}{5}y = 3.$

9. $x + y = 17.$
 $3x + 2y = 44.$

10. $x = 24 - 2y.$
 $3x = 17 - y.$

11. $5x + y = 13.$
 $4x + 3y = 17.$

12. $x - 2y = 25.$
 $3x - 5y = 66.$

13. $x = 5y - 28.$
 $2x = 7y - 35.$

14. $5x - y = 24.$
 $8x - 5y = 1.$

15. $2x + 3y = 17.$
 $3x + 2y = 18.$

16. $x + y = 7.$
 $2x + 5y = 11.$

17. $x - y = 0.$
 $5x - 3y = 14.$

18. $x + 21y = 75.$
 $2x - 33y = 0.$

19. $x + 17y = 300.$
 $11x - y = 104.$

20. $8x + y = 60.$
 $7x - 10y = 9.$

21. $3x - y = 15.$
 $10x + y = -2.$

22. $x + 4y = 24.$
 $x - 13y = -61.$

23. $x = y.$
 $3x + 5y = 120.$

24. $x + 1\frac{2}{3}y = 26\frac{1}{12}.$
 $4\frac{5}{6}y - x = 44\frac{7}{8}.$

25. $x + 40y = 71.$
 $2x - 35y = 27.$

26. $x - y - 1 = 0.$
 $2x + y - 29 = 0.$

III. GENERAL DIRECTIONS

165. The following general directions will be found of some value, although the student must use his judgment in each individual case.

1. *If the equations contain symbols of aggregation, decide whether it is better to remove them at once.*

It is usually best to remove them, as in a case like Ex. 19, p. 166. But in a case like Ex. 18, p. 166, it is evidently better to add at once.

2. *If the equations are in fractional form, decide whether it is better to eliminate without clearing of fractions.*

See p. 158, illustrative problems 4, 5. Much time is often wasted by clearing of fractions unnecessarily. This is also seen in the example on p. 178.

3. *If it seems advisable, clear of fractions and reduce each to the form* $ax + by = c$.

See illustrative problem 1, p. 164. The same course will naturally be followed with an example like Ex. 4, p. 165.

4. *If the coefficient of either unknown quantity is* 1, *it is usually advisable to eliminate by substitution.*

See illustrative problem 1, p. 164, steps 4, 6, 7. This is, however, not often the case.

5. *Otherwise it is generally best to eliminate by addition or subtraction.*

This is the plan usually employed.

In attacking the sets of equations in Exercise LXVII it would be as well to clear of fractions at once in Ex. 1, although it may be solved like Ex. 4, p. 158. In Exs. 2, 3, and 9, it is better not to clear of fractions at first. In Exs. 11 and 12 it is better to clear of fractions at once.

ILLUSTRATIVE PROBLEMS

1. Solve the system of equations

$$1.\quad \frac{1+x}{y}+3=5.$$

$$2.\quad \frac{2}{x}+5=\frac{y}{x}+2.$$

Here it is not best to attempt to eliminate without clearing of fractions. Multiplying both members of (1) by y,

3.	$1+x+3y=5y.$	Ax. 6
4.	$\therefore\ x=2y-1.$	
5.	$2+5x=y+2x$, from (2).	
6.	$\therefore\ 3x-y=-2$, from (5).	
7.	$\therefore\ 6y-3-y=-2$, substituting (4) in (6).	
8.	$\therefore\ 5y=1.$	
9.	$\therefore\ y=\frac{1}{5}$, and $x=\frac{-3}{5}$.	

Check by substituting in *both* given equations.

2. Solve the system of equations

$$1.\quad \frac{1}{x}+y=10.$$

$$2.\quad \frac{3}{x}-y=14.$$

In a case like this it would manifestly be unwise to clear of fractions at once, because it would give a term xy.

3.	$\frac{4}{x}=24$, by adding.
4.	$\therefore\ \frac{1}{x}=6$, by dividing by 4,
5.	$\therefore\ \frac{1}{6}=x$, by multiplying by $\frac{x}{6}$.
6.	$\therefore\ y=4$, by substitution.

EXERCISE LXVII

Solve the following systems of equations:

1. $\frac{x}{a}+\frac{y}{b}=\frac{1}{c}.$
$\frac{x}{m}-\frac{y}{n}=\frac{1}{p}.$

2. $\frac{a}{x}-\frac{b}{y}=c.$
$\frac{m}{x}-\frac{n}{y}=p.$

3. $\frac{11}{y}+\frac{3}{x}=2.$
$\frac{33}{y}-\frac{1}{x}=2\frac{2}{3}.$

4. $\frac{4x+81}{10y-17}=6.$
$\frac{12x+97}{15y-17}=4.$

5. $\frac{3x}{5}+\frac{y}{4}=13.$
$\frac{x}{3}-\frac{y}{8}=3.$

6. $\frac{x}{9}+\frac{y}{7}=6.3.$
$\frac{x}{3}+\frac{53y}{56}=39.2.$

7. $7x-\frac{1}{5}y=48.$
$5y+\frac{1}{7}x=26.$

8. $x+\frac{1}{11}y=71.$
$y-\frac{1}{13}x=61.$

9. $\frac{a}{x}-\frac{b}{y}+1=0.$
$\frac{b^2}{x}+\frac{a^2}{y}=a-b.$

10. $\frac{x+y-1}{x-y+1}=a.$
$\frac{y-x+1}{x-y+1}=ab.$

11. $\frac{x+y+11}{x-y+11}=2.$
$\frac{x-y-11}{x+y-11}=6.$

12. $\frac{x+y+3}{x-y-3}=-1\frac{1}{2}.$
$\frac{x-y-3}{x-y+3}=-2.$

13. $17x-13y=144.$
$23x+19y=890.$

14. $2(x-y)=30x-8y.$
$\frac{x}{3}-y=6x+26.$

15. $\frac{x+y}{9}=\frac{x-y}{15}.$
$\frac{x+3y}{21}=\frac{2x-7y}{3}.$

16. $\frac{x+y}{x-y}=-\frac{15}{8}.$
$9x-\frac{3y+44}{7}=100.$

17. $\frac{4x+2y}{8}=\frac{4y-3x}{5}.$

$\frac{5x-y}{6}=8x-4y+1.$

18. $a(x+y)-b(x-y)=2a.$

$a(x-y)-b(x+y)=2b.$

19. $10[x+9(y-8\overline{x+7})]=6.$

$5[x+4(y-3\overline{x+2})]=1.$

20. $\frac{a}{a+c}x-y=\frac{a-c}{b}-\frac{a}{a+c}y.$

$\frac{x}{c}+\frac{y}{a}=\frac{b}{ac}.$

21. $\frac{5y}{6}-\frac{4y-19}{3}=\frac{x}{6}+\frac{20-2y}{3}.$

$\frac{x+5y}{6}+5=\frac{2y+21}{3}.$

22. $\frac{2x}{y}=\frac{29}{14}.$

$y+4x+6=\frac{4y^2+13xy-12x^2}{4y-3x-1}.$

23. $\frac{13+x}{7}+\frac{3x-8y}{3}=x+y-5\frac{1}{3}.$

$\frac{11-x}{2}+\frac{4x+8y-2}{9}=8-(y-x).$

24. $\frac{4x^2+2xy+288-6y^2}{2x+13-2y}=2x+3y-131.$

$5x-4y=22.$

25. $\frac{7y+13-5x}{4}+y=2x-\frac{3y+2x-16}{3}.$

$x+\frac{5y+2x}{6}-\frac{3x-12+8y}{5}=4-\frac{15+2y-4x}{3}.$

IV. APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES

ILLUSTRATIVE PROBLEMS

1. A certain number of two figures is such that the sum of the digits is 7, and when 27 is added to the number the order of the digits is reversed. Required the number.

1. Let $x =$ the tens' digit, and y the units'.

Then $10x + y =$ the number.

2. Then by the first condition,

$$x + y = 7.$$

3. By the second condition,

$$10x + y + 27 = 10y + x.$$

4. $\therefore x + y = 7,$

$$9x - 9y = -27,$$

or $x - y = -3.$

5. Adding, $2x = 4,$

$$x = 2.$$

$\therefore y = 5$, and the number is 25.

2. The sum of two numbers is 12, and 7 times the quotient of one divided by the other is 5. Required the numbers.

1. Let $x, y =$ the numbers.
2. Then $x + y = 12$, and
3. $7 \cdot \frac{x}{y} = 5$, by the conditions of the problem.
4. $\therefore y = 12 - x$, from (2). Ax. 3
5. And $7x = 5y$, from (3). Ax. 6
6. $\therefore 7x = 60 - 5x$, from (4) and (5).
7. $\therefore 12x = 60$, and $x = 5$. (Why?)
8. $\therefore y = 7$, from step 4.

Check. The sum of 5 and 7 is 12, and 7 times $\frac{5}{7}$ is 5.

EXERCISE LXVIII

1. The sum of two numbers is 30 and their difference is 17. Required the numbers.

2. What is that fraction which equals $\frac{1}{3}$ when 1 is added to the numerator, but equals $\frac{1}{4}$ when 1 is added to the denominator?

3. A number of two figures is 5 times the sum of its digits. If 9 is added to the number, the order of its digits is reversed. Required the number.

4. The sum of two numbers is s and their difference is d. Required the numbers. From the result, deduce a rule for finding two numbers, given their sum and their difference.

5. The sum of two capitals, each invested at 5%, is \$ 12,000, and the sum of 5 yrs. simple interest on the larger and 4 yrs. simple interest on the smaller is \$ 2800. Required the capitals.

6. Divide the two numbers 80 and 90 each into two parts such that the sum of one part of the first and one part of the second shall equal 100, and the difference of the other two parts shall equal 30.

7. Two points move around a circle whose circumference is 100 ft.; when they move in the same direction they are together every 20 secs.; when in opposite directions they meet every 4 secs. Required their rates.

8. A man invests \$ 16,000 for 8 yrs. and \$ 11,000 for 6 yrs., and receives from the two \$ 8090 interest. Had the first been invested at the same rate as the second and the second at the same rate as the first, he would have received \$ 310 more interest in the same times. Required the rate at which each was invested.

9. The sum of two numbers is 49, and if 47 be subtracted from 50 times the first, the result is 3. Required the numbers.

10. Find two numbers the sum of whose reciprocals is 5, and such that the sum of half of the first and one-third of the second equals twice the product of the two numbers.

11. The boat A leaves the city C at 6 A.M.; an hour later the boat B leaves the city D, 80 mi. from C, and meets A at 11 A.M. They would also meet at 11 A.M. if B left at 6 A.M. and A 45 mins. later. Required their rates.

12. Two bodies are 96 yds. apart. If they move towards each other with uniform (but unequal) rates, they will meet in 8 secs.; but if they move in the same direction, the swifter overtakes the slower in 48 secs. Required the rate of each.

13. Of two bars of metal, the first contains 21.875% pure silver and the second 14.0675%. How much of each kind must be taken in order that when melted together the new bar shall weigh 60 oz., and 18.75% shall be pure silver?

14. A reservoir has two contributing canals. If the first is open 10 mins. and the second 13 mins., 15 cu. yds. of water flow in; if the first is open 14 mins. and the second 5 mins., 2.4 cu. yds. more flow in. How many cubic yards of water per minute are admitted by each?

15. A marksman fires at a target 500 yds. distant and hears the bullet strike $4\frac{1}{3}$ secs. after he fires; an observer standing 400 yds. from the target and 650 yds. from the marksman hears the bullet strike $2\frac{1}{3}$ secs. after he hears the report. Required the velocity of sound and the velocity of the bullet, each supposed to be uniform.

ILLUSTRATIVE PROBLEM

Alcohol is received in the laboratory 0.95 pure. How much water must be added to a gallon of this alcohol so that the mixture shall be 0.5 pure?

1. Let $x =$ the *number* of gallons of water to be added.
2. Then $0.5(1 + x)$ represents the alcohol in the mixture.
3. But 0.95 represents the alcohol in the original gallon.
4. $\therefore\ 0.5(1 + x) = 0.95.$
5. $\therefore\ x = 0.9.$

Check. Adding 0.9 gal., there are 1.9 gals. of the mixture, 0.5 of which is the 0.95 gal. of alcohol.

16. How much water must be added to a 5% solution of a certain medicine to reduce it to a 1% solution?

17. How much water must be added to a 15% solution of a certain medicine to reduce it to a 4% solution?

18. How much pure alcohol must be added to a mixture of $\frac{4}{5}$ alcohol so that $\frac{9}{10}$ of the mixture shall be pure alcohol?

19. How much pure alcohol must be added to a mixture of $\frac{3}{4}$ alcohol so that $\frac{7}{8}$ of the mixture shall be pure alcohol?

20. How many ounces of pure silver must be melted with 500 oz. of silver 750 (750 parts pure silver in 1000 parts of metal) fine to make a bar 900 fine?

21. How many ounces of silver 700 fine and how many ounces 900 fine must be melted together to make 78 oz. 750 fine?

22. How many pounds of copper must be melted with 1000 lbs. of gold $\frac{65}{72}$ pure so that the composition shall be 900 fine?

23. How much water must be added to a quart of alcohol 90% pure so that the mixture shall be one quarter pure alcohol?

24. How many ounces of pure silver must be melted with 200 oz. of silver 800 fine to make a bar 900 fine?

25. How much water must be added to a 30% solution of a certain medicine to reduce it to a 15% solution?

26. How much pure alcohol must be added to a gallon of 92% alcohol so that the mixture shall be 93% alcohol?

27. How much water must be added to a pint of alcohol 80% pure so that the mixture shall be two-thirds water?

28. How many ounces of gold must be melted with 10 oz. of gold 14 carats fine ($\frac{14}{24}$ pure) to make an ingot 18 carats fine?

29. How many ounces of copper must be melted with 10 oz. of gold 18 carats fine to make an ingot 14 carats fine?

30. In a certain composition of metal weighing 37.5 lbs., $18\frac{3}{4}$% is pure silver. How many pounds of copper must be melted in so that the composition shall be only 15.625% pure silver?

31. What per cent of the *water* must be evaporated from a 6% solution of salt (salt water which contains 6%, by weight, of salt) so that the remaining portion of the mixture may be a 12% solution?

32. What per cent of the water must be evaporated from a 4% solution of salt so that the remaining portion of the mixture may be a 6% solution?

33. How many pounds of pure water must be added to 32 lbs. of sea water containing 16% (by weight) of salt, in order that the mixture shall contain only 2% of salt?

34. How many pounds of copper should be melted in with 94.5 lbs. of an alloy consisting of 3 lbs. of silver to 4 lbs. of copper so that the new alloy shall consist of 7 lbs. of copper to 2 lbs. of silver?

V. SYSTEMS OF EQUATIONS WITH THREE OR MORE UNKNOWN QUANTITIES

166. In general, *three* linear equations involving *three* unknown quantities admit of determinate values of these quantities. For one of the quantities can be eliminated from the first and second equations, and the same one from the first and third, thus leaving two linear equations involving only two unknown quantities. Similarly for a system of *four* linear equations containing *four* unknown quantities, and so on.

ILLUSTRATIVE PROBLEMS

1. Solve the following system of equations:

1. $5x - 3y + 4z = 17.$
2. $2x + 7y - 5z = 5.$
3. $9x - 2y - z = 8.$

We first proceed to eliminate z from (1) and (2).

4. $25x - 15y + 20z = 85$, multiplying (1) by 5.

5. $8x + 28y - 20z = 20$, multiplying (2) by 4.

6. $\therefore 33x + 13y = 105$, adding (4) and (5).

We now proceed to eliminate z from (1) and (3).

7. $36x - 8y - 4z = 32$, multiplying (3) by 4.

8. $\therefore 41x - 11y = 49$, from (1) and (7).

We now proceed to eliminate y from (6) and (8).

9. $363x + 143y = 1155$, multiplying (6) by 11.

10. $533x - 143y = 637$, multiplying (8) by 13.

11. $\therefore 896x = 1792$, adding.

12. $\therefore x = 2.$

13. $\therefore y = 3$, substituting in (6).

14. $\therefore z = 4$, substituting in (1).

Check. Substitute in (2) and (3). Why not in (1)?

$$4 + 21 - 20 = 5, \text{ and } 18 - 6 - 4 = 8.$$

2. Solve the following system of equations:

$$1.\quad \frac{1}{5x}+\frac{1}{7y}+\frac{1}{9z}=1.$$

$$2.\quad \frac{2}{x}+\frac{3}{y}-\frac{4}{z}=67.$$

$$3.\quad \frac{3}{x}-\frac{5}{y}+\frac{2}{z}=-38.$$

We first proceed to eliminate $\frac{1}{z}$ from (1) and (2).

4. $$\frac{4}{5x}+\frac{4}{7y}+\frac{4}{9z}=4, \text{ from (1).}$$

5. $$\frac{2}{9x}+\frac{3}{9y}-\frac{4}{9z}=\frac{67}{9}, \text{ from (2).}$$

6. $$\therefore \frac{46}{5x}+\frac{57}{7y}=103.$$

We then proceed to eliminate $\frac{1}{z}$ from (2) and (3).

7. $$\frac{6}{x}-\frac{10}{y}+\frac{4}{z}=-76, \text{ from (3).}$$

8. $$\therefore \frac{8}{x}-\frac{7}{y}=-9, \text{ from (2) and (7).}$$

We then proceed to eliminate $\frac{1}{y}$ from (6) and (8).

9. $$\frac{322}{5x}+\frac{399}{7y}=721, \text{ from (6).}$$

10. $$\frac{456}{7x}-\frac{399}{7y}=-\frac{513}{7}, \text{ from (8).}$$

11. $$\frac{4534}{35x}=\frac{4534}{7}, \text{ from (9) and (10).}$$

12. $$\therefore \frac{1}{x}=5, \text{ and } x=\frac{1}{5}.$$

13. $$\therefore \frac{1}{y}=7, \text{ and } y=\frac{1}{7}.$$

14. $$\therefore \frac{1}{z}=-9, \text{ and } z=-\frac{1}{9}.$$

Check. Substitute in (1) and (2). Why not in (3)?

3. Solve the following system of equations:

$$1.\quad w+2x+y-z=4.$$
$$2.\quad 2w+x+y+z=7.$$
$$3.\quad 3w-x+2y-z=1.$$
$$4.\quad 4w+3x-y+2z=13.$$

Eliminating z from (1) and (2),

5. $$3w+3x+2y=11.$$

Also from (1) and (3),

6. $$2w-3x+y=-3.$$

Also from (1) and (4),

7. $$6w+7x+y=21.$$

Eliminating y from (5) and (6),

8. $$w-9x=-17.$$

Also from (6) and (7),

9. $$2w+5x=12.$$

Eliminating w from (8) and (9),

10. $$x=2.$$

$$\therefore\ w-18=-17, \text{ substituting in (8).}$$

11. $$\therefore\ w=1.$$

$$\therefore\ 2-6+y=-3, \text{ substituting in (6).}$$

12. $$\therefore\ y=1.$$

$$\therefore\ 1+4+1-z=4, \text{ substituting in (1).}$$

13. $$\therefore\ z=2.$$

Of course there are various other arrangements of the equations.

Check. Substitute in (2), (3), and (4). Why not in (1)?

$$2+2+1+2=7.$$
$$3-2+2-2=1.$$
$$4+6-1+4=13.$$

EXERCISE LXIX

Solve the following systems of equations:

1. $\frac{1}{x}+\frac{1}{y}=1.$
$\frac{1}{x}+\frac{1}{z}=2.$
$\frac{1}{y}+\frac{1}{z}=\frac{3}{2}.$

2. $\frac{x}{5}+\frac{y}{7}+\frac{z}{9}=258.$
$\frac{x}{7}+\frac{y}{9}+\frac{z}{5}=304.$
$\frac{x}{9}+\frac{y}{5}+\frac{z}{7}=296.$

3. $x+y=16.$
$z+x=22.$
$y+z=28.$

4. $x+2y=23.$
$3x+4z=57.$
$5y+6z=94.$

5. $x=21-4y.$
$z=9-\frac{2}{3}x.$
$y=64-7\frac{1}{2}z.$

6. $x+y+z=10.$
$x-y+z=2.$
$x+y-z=0.$

7. $x+3y=26.$
$3x+2z=37.$
$4y-z=17.$

8. $x+y-z=132.$
$x-y+z=65.4.$
$-x+y+z=-1.2.$

9. $7x-3y=1.$
$11z-7u=1.$
$4z-7y=1.$
$19x-3u=1.$

10. $5x-6y+4z=15.$
$7x+4y-3z=19.$
$x+y=7.$
$x+6z=39.$

11. $2x+y+2z=15.$
$3x+2y+3z=21.$
$3x+3y+4z=27.$

12. $2x+2y+z=23.$
$3x-y+2z=14.$
$x+7y-3z=45.$

13. $x+y+z=5.$
$3x-5y+7z=75.$
$9x-11z+10=0.$

14. $0.2x+0.3y+0.4z=25.$
$0.3x+0.7y+0.6z=45.$
$0.4x+0.8y+0.9z=58.$

15. $\frac{6}{x}+\frac{12}{y}+\frac{18}{z}=18.$
$\frac{12}{x}+\frac{18}{y}+\frac{6}{z}=23.$
$\frac{18}{x}+\frac{6}{y}+\frac{12}{z}=25.$

16. $7x-2z+3u=17.$
$4y-2z+v=11.$
$5y-3x-2u=8.$
$4y-3u+2v=9.$
$3z+8u=33.$

17. $2x+3y=10.$
$3x+4z=14.$
$4y+5z=18.$

18. $41x+2y+z=40.$
$3x+2y-z=0.$
$5x+3y+z=50.$

19. $x+2y+3z=24.$
$2x+3y+4z=34.$
$3x+4y+4z=44.$

20. $3x-5y+4z=0.5.$
$7x+2y-3z=0.2.$
$4x+3y-z=0.7.$

21. $2x-3y+45z=2.$
$5x+y+7z=44.$
$4x-2y+18z=0.$

22. $7x+6y-10z=21.$
$5x-4y+2z=21.$
$2x+8y-7z=21.$

23. $x+y+z=0.$
$2x+3y+4z=-4.$
$3x+5y+3z=0.$

24. $3x+y+z=9.$
$2x-2y-3z=16.$
$5x+y+5z=-25.$

25. $5x+4y+7z=-48.$
$3x-7y-2z=18.$
$2x-y+z=-6.$

26. $x+y+z=3.824.$
$1.25x+23.8y+3.1z=7.5276.$
$1.1x+2y-0.5z=1.8505.$

27. $w-x-y=1.$
$x-y-z=0.$
$y-z-w=3.$
$z-w-x=6.$

28. $w+x+y+z=10.$
$w+2x+y+3z=20.$
$w+3x+2y+z=17.$
$w+4x+3y+2z=26.$

29. $w+x+y=6.$
$x+y+z=6.$
$y+z+w=6.$
$z+w+x=6.$

30. $w+2x+3y+4z=17.$
$2w-x+y-4z=3.$
$3w-x+2y+3z=16.$
$4w+2x+2y+z=27.$

VI. APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES

ILLUSTRATIVE PROBLEM

A certain number of three figures is such that when 198 is added the order of the digits is reversed; the sum of the hundreds' digit and the tens' digit is the units' digit; and the number represented by the two left-hand digits is 4 times the units' digit. Required the number.

1. Let $x =$ the hundreds' digit, y the tens', z the units'.
2. Then $100x + 10y + z =$ the number.
3. Then, by the first condition,

$$100x + 10y + z + 198 = 100z + 10y + x.$$

4. By the second condition, $x + y = z$.
5. By the third condition, $10x + y = 4z$.
6. $\therefore$ the equations are $x - z = -2$, from step 3.

$x + y - z = 0$, from step 4.

$10x + y - 4z = 0$, from step 5.

7. Solving, $x = 1,\ y = 2,\ z = 3.$

$\therefore$ the number is 123.

Check by noting that 123 answers all of the conditions of the *original statement.*

EXERCISE LXX

1. The sum of the digits of a certain number of two figures is 15, and their difference is 1. Required the number.

2. In a company of 118 persons a resolution is carried by a majority of 48, all voting. How many voted for the measure?

3. The sum of two numbers is 47, and the quotient of one divided by the other is 5 with a remainder of 5. Required the numbers.

4. Of two numbers the first is 10 less than 10 times the second, and the second is 2 more than 0.3 the first. Required the numbers.

5. A number of two figures equals 6 times the sum of its digits; the tens' digit is 1 more than the units' digit. Required the number.

6. The difference between two numbers equals the quotient of the larger divided by the smaller, and this equals 5. Required the numbers.

7. The sum of two numbers is 444, and the quotient of one divided by the other is 2 with a remainder of 75. Required the numbers.

8. A father said to his son, "7 yrs. ago I was 7 times as old as you, but 3 yrs. from now I shall be 3 times as old." Required their ages now.

9. Two numbers are such that the first less 3 times the second equals the second less 3 times the first, and their sum is 42. Required the numbers.

10. What three numbers have the peculiarity that the sum of the reciprocals of the first and second is $\frac{1}{2}$, of the first and third $\frac{1}{3}$, and of the second and third $\frac{1}{4}$?

11. If to a certain number of two figures the tens' digit be added, the sum is 100; and if the units' digit be subtracted, the difference is 90. Required the number.

12. The enrolment in a certain class is 30% more than last year, and the sum of the two enrolments is 6 more than twice the number last year. Required the two enrolments.

13. If from twice the first of two numbers the second is taken, the remainder is 63; but if from twice the second the first is taken, the remainder is zero. Required the numbers.

14. Separate the number 96 into three parts such that the first divided by the second gives a quotient 2 and a remainder 3; and the second divided by the third gives a quotient 4 and a remainder 5.

15. Two numbers have the same two digits in reverse order, the first number being 9 more than the second. Twice the tens' digit of the first number equals 3 times the units' digit. Required the numbers.

16. A cistern can be filled by two pipes, A and B, in 35 mins., by A and C in 42 mins., by B and C in 70 mins. How long will it take A, B, and C, all open together, to fill it? How long for each separately?

17. A certain number of three figures is such that its value is unchanged if the order of its digits is reversed. The sum of the digits is 11, and the tens' digit is 1 less than the units' digit. Required the number.

18. The middle digit of a certain number of three figures is half the sum of the other two; the number is 48 times the sum of the digits. Subtracting 198 from the number, the order of the digits is reversed. Required the number.

19. A shell fired against the wind goes 335.94 yds. in a second; if fired with the wind it goes 344.42 yds. in the same length of time. The velocity of the shell being supposed uniform when there is no wind, required this velocity.

20. There is a certain number of six figures, the figure in units' place being 4; if this figure is carried over the other five to occupy the left-hand place, the resulting number is four times the original one. Required the number.

21. A father said to his two sons, one of whom was 4 yrs. older than the other, "In 2 yrs. I shall be twice as old as the two of you together, but 6 yrs. ago I was 6 times as old as the two of you together." Required the ages of the three at present.

22. Separate the number 150 into three parts such that if the third be subtracted from the sum of the other two the remainder is 30, and if the second be subtracted from the sum of the other two the remainder is 50.

23. Separate the number 232 into three parts such that half of the difference between the first and the sum of the other two, a third of the difference between the second and the sum of the other two, and a fourth of the difference between the third and the sum of the other two, are all three equal.

24. Four towns, A, B, C, D, are so situated that the distance from A to D by way of B and C is 48 mi.; from B to A by way of C and D, 51 mi.; from C to B by way of D and A, 50 mi. The distance from C to A by way of D is 4 mi. farther than by way of B. Required the distances AB, BC, CD, DA.

25. The sum of three capitals is $111,000. The first is invested at 4%, the second at $4\frac{1}{2}$%, and the third at 5%, and the total annual interest is $5120. If the first had been invested at $2\frac{1}{2}$%, the second at 3%, and the third at 4%, the total annual interest would have been $3710. Required the capitals.

26. Of three bars of metal, the first contains 750 oz. silver, $62\frac{1}{2}$ oz. copper, $187\frac{1}{2}$ oz. tin; the second, $62\frac{1}{2}$ oz. silver, 750 oz. copper, $187\frac{1}{2}$ oz. tin; and the third no silver, 875 oz. copper, 125 oz. tin. How many ounces from these bars must be melted together to form a bar which shall contain 250 oz. silver, $562\frac{1}{2}$ oz. copper, and $187\frac{1}{2}$ oz. tin?

27. Of three bars of metal, the first contains 750 oz. silver, 200 oz. copper, 50 oz. tin; the second, 800 oz. silver, 125 oz. copper, 75 oz. tin; and the third 700 oz. silver, 250 oz. copper, 50 oz. tin. How many ounces from these bars must be melted together to form a bar which shall contain 765 oz. silver, 175 oz. copper, and 60 oz. tin?

28. There are three numbers such that the sum of any two, less the other, is 9. Required the numbers.

29. There are three numbers such that the third less the second equals the second less the first; the third less the first equals the second; the sum of the numbers is 12. Required the numbers.

30. There are three numbers whose sum is 6, and such that the first and twice the second equals 14 less three times the third, while the sum of the first and five times the third equals four times the remainder after taking the second from 6. Required the numbers.

31. A certain number of three figures is such that if 11 is added to the number made by taking the first two left-hand digits in order, the sum is the number made by taking the second two digits in order. The sum of the first two digits is 6 more than the third digit, and the number is 3 less than 33 times the sum of the digits. Required the number.

32. Two bodies, A and B, start at the same time from the points P and Q, respectively, and move at uniform rates towards one another, B faster than A; at the end of 18 secs., and again at the end of 30 secs., they are 48 ft. apart. Had they moved in the same direction, B following A, at the end of 40 secs. they would have been 48 ft. apart. Determine their rates and the distance PQ.

33. In each of three reservoirs is a certain quantity of water. If 20 gals. are drawn from the first into the second, the second will contain twice as much as the first; but if 30 gals. are drawn from the first into the third, the third will contain 20 gals. less than 4 times as much as the first; but if 25 gals. are drawn from the second into the third, the third will contain 50 gals. less than 3 times the second. How many gallons does each contain?

REVIEW EXERCISE LXXI

1. Divide $a^2 - b^2 - c^2 - 2bc$ by $\dfrac{a+b+c}{a+b-c}$.

2. Factor the expression $63a^2 - 24a - 15$.

3. Find the product of $\dfrac{1-a^2}{1+b}$, $\dfrac{1-b^2}{a^2+a}$, and $1 + \dfrac{a}{1-a}$.

4. Find four factors of $4(ad+bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.

5. Solve the equation $(a+x)(b+x) - a(b+c) = \dfrac{a^2c}{b} + x^2$.

6. Prove that $4xy(x^2 - y^2) = (x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2$.

7. Divide $b(x^3 - a^3) + ax(x^2 - a^2) + a^3(x-a)$ by $(a+b)(x-a)$.

8. Simplify $(a^2 + b^2 + c^2)^2 + (a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.

9. Solve the system of equations

$$10x + 11y = 343.$$
$$7x + 5y = 178.$$

10. A number of two figures has its units' figure twice its tens', and the sum of the digits is 9. Required the number.

11. A number of two figures is 4 times the sum of its digits. If 27 is added to the number, the order of its digits is reversed. Required the number.

12. A certain number of three figures is such that the sum of the units' and hundreds' digits equals the tens' digit, the units' digit exceeds the hundreds' by 1, and the sum of the hundreds' and tens' digits is 8 more than the units' digit. Required the number.

CHAPTER X

INDETERMINATE EQUATIONS

167. A linear equation involving two unknown quantities can be satisfied by any number of values of those quantities.

E.g., in the equation $x + y = 10$,

we have $y = 10 - x$.

Hence if $x = 10$, $y = 0$,

if $x = 9$, $y = 1$,

if $x = 8$, $y = 2$, and so on.

168. Equations like the above, which can be satisfied by an unlimited number of values of the unknown quantities, are called **indeterminate equations.**

169. Since two equations containing three unknown quantities give rise, by eliminating one of these quantities, to a single equation containing only two, it follows that in general, *Two equations, each containing three unknown quantities, are indeterminate as to all of these quantities.*

E.g., the two equations

$$2x + 3y + z = 10,$$
$$3x + 2y + z = 8,$$

give rise to the single equation

$$-x + y = 2,$$

or to $5y + z = 14$,

or to $5x + z = 4$,

all three of which are indeterminate.

170. Roots of an indeterminate equation are often found by simple inspection.

E.g., to find the roots of $2x - 7y = 5$.

Let $x = 0, \quad 1, \quad 2, \quad 3, \ 4, \cdots$

then the corresponding values of y are $\frac{-5}{7}, \ \frac{-3}{7}, \ \frac{-1}{7}, \ \frac{1}{7}, \ \frac{3}{7}, \cdots$.

Similarly, find a set of roots of $x + 2y + 3z = 10$.

Let $z = 1$;

then $x + 2y = 7$;

and if $x = 0, 1, 2, 3, \cdots$

the corresponding values of y are $\frac{7}{2}$, 3, $\frac{5}{2}$, 2, $\cdots$.

That is, the equation is satisfied if

$$z = 1, \ x = 0, \ y = \tfrac{7}{2},$$

or if $z = 1, \ x = 1, \ y = 3$, etc.

Similarly, we may start with $z = 2$.

171. Sometimes it is desirable to find the various *positive integral roots* of an indeterminate equation. For practical purposes these may be found by simple inspection.

E.g., to find the positive integral roots of $5x + 3y = 19$. Here $x \not> 3$, because if $x > 3$, and integral, y is negative.

If $x = 3, \quad 2, \ 1,$

then $y =$ a fraction, 3, a fraction.

$\therefore x = 2, \ y = 3$ are the only positive integral roots of the equation.

Similarly, to find the positive integral roots of $3x + 2y = 25$.

$$\because x = 8\tfrac{1}{3} - \tfrac{2}{3}y, \ x < 8.$$

$$\because y = \tfrac{1}{2}(25 - 3x),$$

x must be odd in order that $25 - 3x$ should be divided by 2.

Hence if $x = 7, 5, 3, 1,$

then $y = 2, 5, 8, 11.$

EXERCISE LXXII

Find three sets of roots of each of equations 1–12.

1. $2x + y = 5.$
2. $x + y = 14.$
3. $4x - 3y = 9.$
4. $3x + 2y = 7.$
5. $4x + 10y = 32.$
6. $7x - y = 11.$
7. $17x + 2y = 42.$
8. $5x - 2y = 17.$
9. $10x + 3y = -4.$
10. $4x + 7y = 20.$
11. $5x + 23y = 100.$
12. $11x - 4y = 17.$

Find all of the positive integral roots of each of equations 13–34.

13. $x + y = 5.$
14. $x + 2y = 9.$
15. $5x + y = 11.$
16. $7x + y = 17.$
17. $7x + y = 23.$
18. $5x + y = 17.$
19. $x + 3y = 17.$
20. $2x + 3y = 6.$
21. $2x + y = 10.$
22. $3x + 2y = 7.$
23. $15x + y = 24.$
24. $4x + 5y = 25.$
25. $2x + 2y = 14.$
26. $2x + 9y = 35.$
27. $3x + 5y = 20.$
28. $5x + 2y = 17.$
29. $4x + 3y = 57.$
30. $2x + 10y = 30.$
31. $20x + 3y = 47.$
32. $10x + 3y = 51.$
33. $20x + 3y = 64.$
34. $15x + 2y = 48.$

Find two entirely different sets of roots of each of equations 35–40.

35. $x - y + z = 4.$
36. $x + 2y + 3z = 6.$
37. $x + y + z = 10.$
38. $x - 3y + 4z = 20.$
39. $2x + 10y - z = 15.$
40. $3x - 7y + 5z = 12.$

REVIEW EXERCISE LXXIII

1. Multiply $(a^2 - 2a + 2)^2$ by $a^2 + 6a + 1$.

2. Simplify $x - [5y - (x - \overline{3z - 3y} + 22 - \overline{x - 2y - z})]$.

3. Divide the product of $x^2 + xy + y^2$ and $x^3 + y^3$ by $x^4 + x^2y^2 + y^4$.

4. Prove that $(a + b)(b + c)(c + a) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$.

5. If $x = y = a$, show that $x^2 + 8xz + y^2 + 4z^2 + 2xy$ then equals $4(x + z)^2$.

6. Simplify $\left(\dfrac{x + y}{x - y} + \dfrac{x - y}{x + y}\right) \div \left(\dfrac{x^2 + y^2}{x^2 - y^2} - \dfrac{x^2 - y^2}{x^2 + y^2}\right)$.

7. Solve the equation

$$x - 3 - (3 - x)(1 + x) = (x - 3)(x + 1) + 3 - x.$$

8. Solve the equations $\dfrac{x}{m + n} + \dfrac{y}{m - n} = 2m$, $\dfrac{x - y}{4mn} = 1$.

9. Solve in positive integers the equation $2x + 3y = 21$.

10. Solve in positive integers the equation $5x + 3y = 20$.

11. Find the highest common factor of $a^3 + 1$ and $a^3 + ba^2 + ba + 1$.

12. Find the lowest common multiple of $6(x^3 - y^3)(x - y)^3$, $9(x^4 - y^4)(x - y)^2$, and $12(x^2 - y^2)^3$.

13. At what time between 1 and 2 o'clock is the minute hand of a clock a minute space ahead of the hour hand?

14. A and B run a mile race. A gives B a start of 44 yds. and beats him by 51 secs.; in a second race, A gives B a start of 1 min. 15 secs. and is beaten by 88 yds. Find the rates per mile of A and B separately.

CHAPTER XI

INVOLUTION AND EVOLUTION

I. INVOLUTION

172. **The product of several equal factors is called a power** of one of them (§ 8).

The broader meaning of the word *power* is discussed later (§ 199). At present the term will be restricted to positive integral power.

173. The operation of finding a power of a number or of an algebraic expression is called **involution.**

The student has already proved one important proposition in involution, viz., that $a^m \cdot a^n = a^{m+n}$, where the exponents are positive integers (§ 60).

He has also learned how to raise the binomial $x \pm y$ to the second and third powers (§ 68).

It now becomes necessary to consider certain other theorems.

174. **Notation.** If m and n are positive integers,

$(a^m)^n$ means $a^m \cdot a^m \cdot a^m \cdots$ to n factors, each a^m;

a^{m^n} " $a \cdot a \cdot a \cdots$ to m^n " " a.

E.g., $(a^3)^2$ means $a^3 \cdot a^3 = a^{3+3} = a^6$;

a^{3^2} " $a \cdot a \cdot a \cdots$ to 3^2 factors, $= a^9$;

a^{2^3} " $a \cdot a \cdot a \cdots$ to 2^3 " $= a^8$,

175. a^1 has already been defined to equal a (§ 9).

176. The expression a^0, a being either positive or negative, is defined to equal 1, for reasons hereafter set forth (§ 200).

177. There are three important laws of exponents in involution. Of these the first is

$$(a^m)^n = a^{mn}.$$

This is easily seen to be true in the special case of $(a^3)^2$;

for $\qquad (a^3)^2$ means $a^3 \cdot a^3 = a^{3+3} = a^6$. $\qquad$ § 174

That it is true in general appears from the following:

$$\begin{aligned} (a^m)^n \text{ means } & a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors} \\ &= a^{m+m+m+\cdots \text{ to } n \text{ terms}} \qquad \text{§ 174} \\ &= a^{mn}. \end{aligned}$$

Hence, in general,

The n*th power of the* m*th power of an algebraic expression equals the* mn*th power of the expression.*

178. The second important law is

$$(abc \cdots)^m = a^m b^m c^m \cdots.$$

This is easily seen to be true in the special case of $(ab)^2$;

for $\qquad (ab)^2$ means $ab \cdot ab = a \cdot a \cdot b \cdot b = a^2 b^2$.

That it is true in general appears from the following:

$$\begin{aligned} &(abc \cdots)^m \text{ means } (abc \cdots) \cdot (abc \cdots) \cdot (abc \cdots) \cdots, \\ &\text{to } m \text{ groups, each } (abc \cdots) \\ &= (aaa \cdots \text{ to } m \text{ factors}) \cdot (bbb \cdots \text{ to } m \text{ factors}) \cdots \\ &= a^m b^m c^m \cdots. \end{aligned}$$

Hence, in general,

The m*th power of the product of several algebraic expressions equals the product of the* m*th powers of the expressions.*

179. **The third important law is**

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$$

This is evidently true, for

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots n \text{ factors},$$

$$=\frac{aaa\cdots n \text{ factors}}{bbb\cdots n \text{ factors}},$$

$$=\frac{a^n}{b^n}.$$

Hence, in general,

The n*th power of a fraction equals the* n*th power of the numerator divided by the* n*th power of the denominator.*

180. **Law of signs.** Since

$$+a\cdot+a=+a^2,$$

and $$-a\cdot-a=+a^2,$$

but $$-a\cdot-a\cdot-a=-a^3,$$

it is easily seen that

1. *All powers of positive expressions are positive;*
2. *Even powers of negative expressions are positive;*
3. *Odd powers of negative expressions are negative.*

ILLUSTRATIVE PROBLEMS

1. Expand $(-a^2b^3c^4)^3$.

The power is negative. § 180

It is the product of the 3d powers of the factors a^2, b^3, c^4. § 178

Hence it is $-a^6b^9c^{12}$.

2. Expand $\left(\frac{a^2b}{cd^2}\right)^3$ By §§ 177–179, this equals $\frac{a^6b^3}{c^3d^6}$.

EXERCISE LXXIV

Expand the following expressions:

1. $(a^2b)^4$.
2. $(abc^2)^6$.
3. $(a^mb^n)^2$.
4. $(a^2x^m)^n$.
5. $(x^mb^m)^m$.
6. $(a^2b^3c^4d^5)^2$.
7. $(-a^2b^3c)^3$.
8. $(-ab^2c^3)^4$.
9. $-(a^2b^3c^4)^4$.
10. $(a^3)^4$, $(a^4)^3$.
11. $(a^2)^5$, $(a^5)^2$.
12. $(a^m)^n$, $(a^n)^m$.
13. $(-a^2xy^2z)^3$.
14. $(-x^2y^3z^9)^7$.
15. $(-\frac{1}{2}a^2b^m)^{4m}$.
16. $-[-(a^2)^2]^2$.
17. $-(-a^mb^nc^p)^2$
18. $(-ab^2x^4y^5z^6)^4$.
19. $(-p^2q^3r^4s^5)^5$.
20. $(-m^2n^4p^2q^4)^6$.
21. $(-a^6b^5c^4d^2e)^7$.
22. $\left(\frac{a^2b}{c^2d^3}\right)^2$.
23. $\left(\frac{xyz^2}{pqr^2}\right)^5$.
24. $-\left(\frac{a^mb^n}{x^ny^m}\right)^2$.
25. $\left(-\frac{a^mb^n}{x^py^q}\right)^2$.
26. $\left(\frac{2\,a^3bc}{3\,x^2yz}\right)^3$.
27. $\left(\frac{3\,a^{m-1}b}{2\,x^{m-1}y}\right)^5$.
28. $\left(-\frac{m^2n^3}{p^3q^2}\right)^3$.
29. $\left(-\frac{a^2b^4}{m^4n^6}\right)^4$.
30. $\left(-\frac{a^{m-n}b}{x^{n-m}y}\right)^3$.
31. $\left(\frac{5\,x^2y^3z^4}{3\,ab^2c^3}\right)^3$.
32. $-\left(\frac{a^nb^mc^s}{-n^amb}\right)^2$.
33. $\left(-\frac{x^2y^3z^4}{mnp}\right)^6$.
34. $-\left(\frac{a^{2m}b^{3m}}{p^{4m}}\right)^{2n}$.
35. $\left(-\frac{a^{m-1}b}{ax^2y}\right)^2$.
36. $\left(\frac{a^2b^3b^4}{-x^2y^3z^4}\right)^3$.
37. $\left(\frac{p^nq^nr^n}{-a^nb^nc^n}\right)^2$.
38. $\left(-\frac{abc^2d^3}{p^2q^3r^4}\right)^4$.
39. $\left(\frac{-6\,m^np^q}{9\,n^mq^p}\right)^2$.
40. $\left(-\frac{abx^2y^3}{cdm^4n^5}\right)^5$.
41. $\left(-\frac{a^{2m}b^{3m}}{p^{4m}}\right)^{2n}$.
42. $\left(\frac{u^2v^3w^4x^5y^6z^7}{ab^2c^3d^4e^5}\right)^3$.
43. $-\left(\frac{-a\cdot -b\cdot -c}{-m\cdot -n\cdot -p}\right)^3$.
44. $\left(\frac{-a^2\cdot -b^2\cdot -c^2}{-x\cdot y\cdot -z}\right)^2$.

181. Powers of polynomials. A polynomial can be raised to any power by ordinary multiplication.

But in raising to the 4th power it is easier to square and then to square again, since $(a^2)^2 = a^4$.

E.g., to expand $(x-2y)^4$.

1. $(x-2y)^2 = x^2 - 4xy + 4y^2$. § 68

2. $(x^2 - 4xy + 4y^2)^2 = (x^2 - 4xy)^2 + 2(x^2 - 4xy)\cdot 4y^2 + 16y^4$

3. $\qquad = x^4 - 8x^3y + 16x^2y^2 + 8x^2y^2 - 32xy^3 + 16y^4$

4. $\qquad = x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.

Similarly, to raise to the 6th power, first cube and then square, since $(a^3)^2 = a^6$. But to raise to the 5th, 7th, ... powers multiply the expressions.

EXERCISE LXXV

Expand the following expressions:

1. $(x+y)^5$.
2. $(x+y)^6$.
3. $(20+1)^2$.
4. $(2a+1)^4$.
5. $(x+3y)^2$.
6. $(x-3y)^3$.
7. $(x+2y)^5$.
8. $(x-2y)^6$.
9. $(2a+3)^4$.
10. $(1-2a)^4$.
11. $(a^{10}-b^8)^2$.
12. $(a+3b)^3$.
13. $(x^m+y^n)^2$.
14. $(2a-3)^4$.
15. $(a+2b^2)^3$.
16. $(-a-b)^3$.
17. $(x^2-3y^3)^2$.
18. $(a^2+2ab)^3$.
19. $(2x-7y)^2$.
20. $(a-b-c)^4$.
21. $(a+b+c)^4$.
22. $(a-b+c)^3$.
23. $(-x-3y)^3$.
24. $(2x^3-3y^4)^3$.
25. $(a+2b+c)^2$.
26. $(m^2+m+1)^2$.
27. $(m^2-m-1)^2$.
28. $(x^4+x^2y^2+y^4)^2$.

182. The Binomial Theorem. It frequently becomes necessary to raise binomials to various powers. There is a simple law for effecting this, known as the *Binomial Theorem.*

A complete proof of this theorem, for positive exponents, is given later in this work. For the present it is merely necessary for the student to discover the general law.

By multiplication it is easily shown that

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

183. Questions on the expansions in § 182.

1. How do the successive exponents of b change?

2. How do the successive exponents of a change?

3. How does the number of terms in each expansion compare with the degree of the binomial?

4. What is the first coefficient? How does the second coefficient compare with the exponent of the binomial?

5. In the case of the 4th power the third coefficient equals $\frac{4 \cdot 3}{2}$. In the 5th power it is $\frac{5 \cdot 4}{2}$. What will it probably be in the 6th power? in the 7th? in the nth?

6. In the case of the 4th power the fourth coefficient equals $\frac{4 \cdot 3 \cdot 2}{2 \cdot 3}$. In the 5th power it is $\frac{5 \cdot 4 \cdot 3}{2 \cdot 3}$. What will it probably be in the 6th power? in the 7th? in the nth?

7. In the case of the 4th power the fifth coefficient equals $\frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 4}$. In the 5th power it is $\frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}$. What will it probably be in the 6th power? in the nth?

8. In expanding $(a+b)^6$, what will be the coefficient of a^5b? of a^4b^2? (Answer without actual multiplication.)

184. From the answers in § 183 it appears that if the binomial $a+b$ is raised to the nth power, n integral and positive, the result is expressed by the formula

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots,$$

where:

1. *The number of terms in the second member is* n + 1;

2. *The exponents of* a *decrease by* 1 *from* n *to* 0, *while those of* b *increase by* 1 *from* 0 *to* n;

3. *The first coefficient is* 1, *the second is* n, *and any other is formed by multiplying the coefficient of the preceding term by the exponent of* a *in that term and dividing by* 1 *more than the exponent of* b.

ILLUSTRATIVE PROBLEMS

1. Expand $(2a-3b^2)^3$.

$$(2a-3b^2)^3 = (2a)^3 + 3(2a)^2(-3b^2) + 3(2a)(-3b^2)^2 + (-3b^2)^3$$
$$= 8a^3 - 36a^2b^2 + 54ab^4 - 27b^6.$$

Check. $(-1)^3 = 8 - 36 + 54 - 27 = -1$.

In cases like this it is better to indicate the work in the first step and then simplify.

2. Expand $(x^2-2y)^5$.

$$(x^2-2y)^5 = (x^2)^5 - 5(x^2)^4\cdot 2y + 10(x^2)^3\cdot(2y)^2 - 10(x^2)^2\cdot(2y)^3 + 5(x^2)\cdot(2y)^4 - (2y)^5$$
$$= x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5.$$

3. Expand $(a+b-c)^2$.

$$(a+b-c)^2 = (\overline{a+b}-c)^2 = (a+b)^2 - 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 - 2ac - 2bc + c^2.$$

EXERCISE LXXVI

Expand the following expressions:

1. $(x+y)^4$.
2. $(1-a)^8$.
3. $(x^2-y)^4$.
4. $(x-y)^5$.
5. $(x^3+y)^6$.
6. $(x^2-y)^8$.
7. $(a-2b)^2$.
8. $(3x-2)^5$.
9. $(2x+3)^3$.
10. $(2x-1)^8$.
11. $(2x-1)^5$.
12. $(2x^2-5)^3$.
13. $(2x-1)^7$.
14. $(3x^3-1)^4$.
15. $(2x^2-3)^4$.
16. $(2x+y^3)^2$.
17. $(a-b+c)^2$.
18. $(2a-3b)^4$.
19. $(a-b+c)^3$.
20. $(x+y-z)^2$.
21. $(x^2y-3y^2)^2$.
22. $(3x+2y^2)^5$.
23. $(a-b-c)^2$.
24. $(a^2-b+c)^2$.
25. $(\frac{1}{2}x^2-\frac{1}{3}y^2)^3$.
26. $(x^3+y^3+z^3)^3$.
27. $(a^2+b^3+1)^2$.
28. $(2a^2+a+1)^2$.
29. $(a-b+2c)^3$.
30. $(2a+b^2+1)^3$.
31. $(2a-b-1)^2$.
32. $(3a^2-2ab+b^2)^3$.
33. $\left(\frac{a+b}{c+d}\right)^2$.
34. $\left(1-\frac{1}{x}\right)^6$.
35. $\left(x+\frac{1}{x}\right)^2$.
36. $\left(\frac{a-b}{a+b}\right)^4$.
37. $\left(\frac{a^2+1}{a^2-1}\right)^4$.
38. $\left(\frac{1}{x^2}-3x^3\right)^3$.
39. $\left(\frac{1}{2}x^3+\frac{1}{y}\right)^2$.
40. $\left(\frac{a+b+1}{a+b-1}\right)^2$.
41. $\left(\frac{a^2-b-c}{a^2-b+c}\right)^2$.
42. $\left(\frac{4m^3}{3n^2}-\frac{2n}{3m^4}\right)^3$.

II. EVOLUTION

185. If an algebraic expression is the product of two equal factors, one of those factors is called the **square root** of the expression. Similarly, one of three equal factors is called the **cube root**, one of four equal factors the **4th root**, ..., one of n equal factors the **nth root**.

The broader meaning of the word *root* is discussed later (§ 194).

186. The process of finding a root of an algebraic quantity is called **evolution**.

Evolution is, therefore, a particular case of factoring. It is evidently the inverse of Involution, as Root is the inverse of Power.

Thus, to find the cube root of 1728 is merely to separate 1728 into three equal factors. This may be done as follows:

$$\begin{aligned}1728 &= 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\\ &= 2\cdot 2\cdot 3\times 2\cdot 2\cdot 3\times 2\cdot 2\cdot 3\\ &= 12^3. \quad \text{Hence 12 is the cube root.}\end{aligned}$$

187. Symbolism. Square root is indicated either by the fractional exponent $\frac{1}{2}$ or by the old radical sign $\sqrt{\ }$, a form of the letter r, the initial of the Latin *radix* (root).

Similarly, $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$ means the cube root of a,

$a^{\frac{1}{4}}$ " $\sqrt[4]{a}$ " " 4th " "

and, in general, $a^{\frac{1}{n}}$ " $\sqrt[n]{a}$ " " nth " "

For present purposes it is immaterial which set of symbols is used. The student should, however, accustom himself to the fractional exponent, which, while a little more difficult to write, has many advantages over the older radical sign, as will be seen later.

188. Law of signs. Since any power of a positive quantity is positive, but even powers of a negative quantity are positive while odd powers are negative (§ 180), therefore,

1. *An even root of a positive quantity is either positive or negative.*

E.g., $4^{\frac{1}{2}} = \pm 2$, $81^{\frac{1}{4}} = \pm 3$.

2. *An odd root of any quantity has the same sign as the quantity itself.*

E.g., $8^{\frac{1}{3}} = 2$, $(-8)^{\frac{1}{3}} = -2$.

3. *An even root of a negative quantity is neither a positive nor a negative quantity.*

E.g., $\sqrt{-1}$ is neither $+1$ nor -1.

189. An even root of a negative quantity is said to be *imaginary*, and imaginary quantities are discussed later (Chap. XIII).

190. The root of a monomial power is easily found by inspection.

E.g.,
$$\because 4a^2b^4 = 2 \cdot 2 \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b,$$
$$\therefore \sqrt{4a^2b^4} = \sqrt{(2 \cdot a \cdot b \cdot b) \cdot (2 \cdot a \cdot b \cdot b)} = \pm 2 \cdot a \cdot b \cdot b$$
$$= \pm 2ab^2.$$

Similarly, $\sqrt[3]{64x^3y^6} = 4xy^2$,
$$\sqrt[5]{32x^{15}y^{30}} = 2x^3y^6,$$
and $\sqrt[6]{64x^{12}} = \pm 2x^2$.

191. It is therefore evident that to extract the root of a monomial power it is merely necessary to *divide the exponent of each letter by the index of the root, extract the root of the numerical coefficient, and prefix the proper sign.*

EXERCISE LXXVII

Simplify the following expressions:

1. $\sqrt{4\sqrt{a^8}}$.

2. $\sqrt[3]{-8a^6b^{12}}$.

3. $\sqrt[7]{x^{70}y^{63}z^{42}}$.

4. $\sqrt[7]{-a^{14}b^{21}c^{35}}$.

5. $\sqrt[5]{243x^{10}y^{15}z^{60}}$.

6. $\sqrt[3]{125x^{15}y^{30}z^{27}}$.

7. $\sqrt[4]{16a^8b^{16}c^{32}d^{40}}$.

8. $\sqrt[3]{3^3(a-2b)^6}$.

9. $\sqrt[3]{-64a^{15}b^{18}c^9}$.

10. $\sqrt[3]{216m^9n^6p^3q^{12}}$.

11. $\sqrt[5]{-32m^5n^{10}p^{15}}$.

12. $-\sqrt[3]{-\sqrt{x^6y^{12}z^{24}}}$.

13. $\sqrt{16a^{4m}\sqrt[3]{a^{12n}b^{6p}}}$.

14. $-\sqrt[3]{-a^{21}b^{33}c^{45}d^{48}}$.

15. $\sqrt{\dfrac{a^2b^4}{m^6n^8}}$.

16. $\sqrt[3]{\dfrac{27a^6b^9c^{12}}{8m^{12}n^9p^6}}$.

17. $\sqrt{2a^2\sqrt{2b^4\sqrt{4c^8}}}$.

18. $\sqrt[9]{x^9y^{18}z^{27}}-\sqrt[6]{x^6y^{12}z^{18}}$.

19. $\sqrt[10]{a^{10}b^{20}c^{30}}+\sqrt[4]{a^4b^8c^{12}}$.

20. $\sqrt[6]{64x^{12}y^{18}}+\sqrt[3]{27x^6y^9}$.

21. $\sqrt[5]{32a^{10}b^{15}}-\sqrt[4]{16a^8b^{12}}$.

22. $\sqrt[5]{32p^{10}q^{20}r^5}+\sqrt[7]{p^{14}q^{28}r^7}$.

23. $\sqrt{144x^{16}y^{20}}+\sqrt[3]{64x^{24}y^{30}}$.

24. $\sqrt[3]{1728a^9b^6}+\sqrt{144a^{12}b^4}$.

25. $\sqrt[4]{81m^8n^{16}}+\sqrt[3]{125m^6n^{12}}$.

26. $\sqrt{16x^{10}y^{20}z^{30}}$, $\sqrt[5]{32x^{10}y^{20}z^{30}}$.

27. $\sqrt[2x]{x^{2x}}$, $\sqrt[2x+1]{x^{2x+1}}$, $\sqrt[2x+1]{-x^{2x+1}}$.

28. $\sqrt{64x^{18}y^{12}}$, $\sqrt[3]{64x^{18}y^{12}}$, $\sqrt[6]{64x^{18}y^{12}}$.

29. $\sqrt{729a^{18}b^6}$, $\sqrt[3]{729a^{18}b^6}$, $\sqrt[6]{729a^{18}b^6}$.

30. $\sqrt[m]{a^{2m}b^{3m}}$, m being even; m being odd.

192. Roots extracted by inspection. The roots of the monomials given on p. 196 were extracted by inspection. Similarly, the roots of polynomials can often be found by inspection.

ILLUSTRATIVE PROBLEMS

1. What is the square root of $x^4 + 4x^2y + 4y^2$?

1. $\because [\pm(f+n)]^2 = f^2 + 2fn + n^2$, §§ 68, 182
2. and $\because$ this polynomial can be arranged in a similar form, viz.,

$$(x^2)^2 + 2x^2(2y) + (2y)^2,$$

3. $\therefore$ it is evidently the square of $\pm(x^2 + 2y)$.

2. Find the cube root of $x^6 + 6x^4y + 12x^2y^2 + 8y^3$.

1. $\because (f+n)^3 = f^3 + 3f^2n + 3fn^2 + n^3$, §§ 68, 182
2. and $\because$ this polynomial can be arranged in a similar form, viz.,

$$(x^2)^3 + 3(x^2)^2 \cdot 2y + 3x^2(2y)^2 + (2y)^3,$$

3. $\therefore$ it is evidently the cube of $x^2 + 2y$.

EXERCISE LXXVIII

Extract the square roots in Exs. 1–10.

1. $4x^2 + 4x + 1$.
2. $4x^4 - 4x^2 + 1$.
3. $9x^6 - 6x^3 + 1$.
4. $\frac{4}{9}x^8 - \frac{4}{5}x^4 + \frac{9}{25}$.
5. $1 - 10x^3 + 25x^6$.
6. $1 + 14x^4 + 49x^8$.
7. $4a^2 - 12ab + 9b^2$.
8. $4x^2 - 12xy + 9y^2$.
9. $4m^2 - 12mx^2 + 9x^4$.
10. $9a^6 - 30a^5 + 25a^4$.

Extract the cube roots in Exs. 11–16.

11. $8x^3 - 12x^2 + 6x - 1$.
12. $27x^3 - 27x^2 + 9x - 1$.
13. $x^3 - 15x^2 + 75x - 125$.
14. $27x^3 + 54x^2 + 36x + 8$.
15. $m^6 + 6m^4n + 12m^2n^2 + 8n^3$.
16. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

193. Square root by the formula $f^2 + 2fn + n^2$. The subject is best understood by following the solution of a problem.

ILLUSTRATIVE PROBLEMS

1. Required the square root of $4x^4 - 12x^2y + 9y^2$.

Let $f =$ the found part of the root at any stage of the operation, and

$n =$ the next term to be found.

Then $(f + n)^2 = f^2 + 2fn + n^2$. § 182

The work may be arranged as follows:

$$\begin{array}{rlll}
\text{Root} & = \pm(2x^2 - 3y) & & \\
\text{Power} = & 4x^4 - 12x^2y + 9y^2 & \text{contains} & f^2 + 2fn + n^2 \\
f^2 = & \underline{4x^4 \qquad\qquad\qquad} & & \\
2f = 4x^2 & \quad -12x^2y + 9y^2 & `` & 2fn + n^2 \\
\underline{2f + n = 4x^2 - 3y} & \underline{\quad -12x^2y + 9y^2} & = & ``
\end{array}$$

EXPLANATION. 1. If a root is arranged according to the powers of some letter, the square obtained by ordinary multiplication will be so arranged.

2. $\therefore$ the square is arranged according to the powers of x, so that the square root of the first term shall be the first term of the root.

3. $\because 4x^4 =$ the square of the first term, the first term is $2x^2$.

4. Subtracting f^2, the remainder, $-12x^2y + 9y^2$, contains $2fn + n^2$.

5. Dividing $2fn$ (*i.e.*, $-12x^2y$) by $2f$ (*i.e.*, $4x^2$), n is found to be $-3y$.

6. $\because f^2 = 4x^4$, and $2fn + n^2 = -12x^2y + 9y^4$, $\therefore$ the sum of these is the square of $\pm(2x^2 - 3y)$.

Check. Let $x = y = 1$. Then $(-1)^2 = 4 - 12 + 9 = 1$.

We might, after a little practice, detach the coefficients, but this is not of much advantage to beginners. The above problem would appear as follows:

$$\begin{array}{lll}
 & 4 - 12 + 9 & \underline{|2 - 3} \\
 & \underline{4 \qquad\qquad} & \\
4 & \quad -12 + 9 & \\
\underline{4 - 3} & \underline{\quad -12 + 9} & \pm(2x^2 - 3y).
\end{array}$$

2. Required the square root of

$$a^2 - 2\,ab^2 + b^4 + 4\,ac - 4\,b^2c + 4\,c^2.$$

	Root $= \pm(a - b^2 \quad +2\,c)$			
	Power $= a^2-2\,ab^2+b^4+4\,ac-4\,b^2c+4\,c^2$		contains	$f^2+2fn+n^2$
f^2	$= a^2$			
$2f$	$=2\,a$	$-2\,ab^2+b^4+\cdots$	"	$2fn+n^2$
$2f+n$	$=2\,a-b^2$	$-2\,ab^2+b^4$	$=$	"
$2f$	$=2\,a-2\,b^2$	$4\,ac-4\,b^2c+4\,c^2$	contains	$2fn+n^2$
$2f+n$	$=2\,a-2\,b^2+2\,c$	$4\,ac-4\,b^2c+4\,c^2$	$=$	"

EXPLANATION. 1. See p. 199 for explanation down to $2f=2\,a-2\,b^2$.

2. $\because f^2 = a^2$, and

$$2fn + n^2 = -2\,ab^2 + b^4,$$

$\therefore (f+n)^2 = a^2 - 2\,ab^2 + b^4$, the square of $a - b^2$.

3. $\because a - b^2$ has now been found, it may be designated by f.

4. $\therefore 4\,ac - 4\,b^2c + 4\,c^2$ contains $2fn + n^2$, the square of $a - b^2$ having been subtracted.

Check. Let $a = b = c = 1$. Then $2^2 = 1 - 2 + 1 + 4 - 4 + 4 = 4$.

EXERCISE LXXIX

Extract the square roots of the following expressions:

1. $4\,a^{16} + 4\,a^8 + 1.$
2. $x^4 + 2\,x^3 - x + \frac{1}{4}.$
3. $\dfrac{a^4}{9} + \dfrac{4\,a^3}{3} + 2\,a^2 - 12\,a + 9.$
4. $\dfrac{x^4}{16} + \dfrac{x^3y}{4} + \dfrac{3\,x^2y^2}{4} + xy^3 + y^4.$
5. $1 + 8\,a + 22\,a^2 + 24\,a^3 + 9\,a^4.$
6. $9\,p^4 - 12\,p^3 + 40\,p^2 - 24\,p + 36.$
7. $9(a^2 - 1)^2 - 12(a^2 - 1)\,a + 4\,a^2.$
8. $4\,x^4 + 4\,x^3y - 7\,x^2y^2 - 4\,xy^3 + 4\,y^4.$
9. $p^2 + 2\,pq + q^2 - 2\,pr - 2\,qr + r^2.$
10. $x^6 - 6\,x^4 + 4\,x^3 + 9\,x^2 - 12\,x + 4.$

11. $1 - 2q - 6p + q^2 + 6pq + 9p^2$.

12. $x^6 + 2x^5y + 3x^4y^2 - x^2y^4 - 2xy^5 + y^6$.

13. $49a^4 - 28a^3b - 17a^2b^2 + 6ab^3 + \frac{9b^4}{4}$.

14. $m^4 - 6m^2n^2 + 25n^4 - 20mn^3 + 4m^3n$.

15. $x^6 - 2ax^5 + a^2x^4 - 2bx^3 + 2abx^2 + b^2$.

16. $\frac{m^4}{n^4} + \frac{9n^4}{m^2} - \frac{4m^3}{n} - 12n^3 + 4m^2n^2 + 6m$.

17. $25a^2 + 9b^2 + c^2 + 6bc - 10ca - 30ab$.

18. $10x^4 - 10x^3 - 12x^5 + 5x^2 + 9x^6 - 2x + 1$.

19. $9x^8 - 12ax^7 + 4a^2x^6 + 6a^3x^5 - 4a^4x^4 + a^6x^2$.

20. $16 - 8m - 23m^2 + 22m^3 + 5m^4 - 12m^5 + 4m^6$.

21. $9a^6b^4 - 12a^4b^5 + 4a^2b^6 + 24a^5b^3c^3 - 16a^3b^4c^3 + 16a^4b^2c^6$.

194. Extension of the definition of root. If an algebraic expression is not the product of r equal factors, it is still said to have an rth root. In such a case **the rth root to n terms** is defined to be that polynomial of n terms found by proceeding as in the ordinary method of extracting the rth root of a perfect rth power.

E.g., the square root of $1 - x$ to 5 terms is

$$\pm(1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \cdots).$$

In the same way we may speak of the roots of numbers which are not perfect powers. Thus the square root of 2 to two decimal places is 1.41.

EXERCISE LXXX

Extract the square roots of the following expressions:

1. $1 + x$ to 4 terms.
2. $4 + 2x$ to 4 terms.
3. $x^2 + y$ to 3 terms.
4. $x^2 + ax$ to 3 terms.
5. $x^2 + 1$ to 3 terms.
6. $9a^2 + 12ax$ to 2 terms.
7. $1 - 2x$ to 4 terms.
8. $16x^6 - 24x^5$ to 3 terms.

195. The square roots of numbers are similarly found.

Required the square root of 547.56.

$$\begin{array}{llrcll}
 & & \text{Root} = 2\;\; 3.\;\; 4 & & & \\
 & & \text{Power} = 5\,47.56 & \text{contains} & f^2 + 2fn + n^2 & \\
 & & f^2 = 4\,00.00 & & & \\
\hline
2f & = 40 & 1\,47.56 & \text{“} & 2fn + n^2 & f = 20 \\
2f + n & = 43 & 1\,29.00 & = & \text{“} & n = \;\;3 \\
\hline
2f & = 46 & 18.56 & \text{contains} & 2fn + n^2 & f = 23 \\
2f + n & = 46.4 & 18.56 & = & \text{“} & n = \;\;0.4 \\
\hline
\end{array}$$

Explanation. 1. $\because$ the highest order of the power is 100's, the highest order of the root is 10's, and it is unnecessary to look below 100's for the square of 10's.

2. The greatest square in the 100's is 400, which is the square of 20, which may be called f, the first found part.

3. Subtracting, 147.56 contains $2fn + n^2$, because f^2 has been subtracted from $f^2 + 2fn + n^2$, where f stands always for the *found* part and n for the *next order* of the root.

4. $2fn + n^2$ is approximately the product of $2f$ and n, and hence, if divided by $2f$, the quotient is approximately n. $\therefore n = 3$.

5. $\therefore 2f + n = 43$, and this, multiplied by n, equals $2fn + n^2$.

6. $\because f^2$ has already been subtracted, after subtracting $2fn + n^2$ there has been subtracted $f^2 + 2fn + n^2$, or $(f + n)^2$, or 23^2.

7. Calling 23 the second found part, f, it appears that 23^2, or f^2, has been subtracted.

8. $\therefore$ the remainder 18.56 contains $2fn + n^2$.

9. Dividing by $2f$ for the reason already given, $n = 0.4$.

10. $\therefore 2f + n = 46.4$, and $18.56 = 2fn + n^2$, as before.

EXERCISE LXXXI

Extract the square roots of the following numbers:

1. 958441.	2. 7779.24.	3. 32.6041.
4. 24.1081.	5. 0.900601.	6. 0.055696.
7. 1225.	8. 1681.	9. 23409.

196. Cube root by the formula $f^3 + 3f^2n + 3fn^2 + n^3$.

Required the cube root of $8a^3 - 12a^2b + 6ab^2 - b^3$.

Let $f =$ the found part of the root at any stage of the operation, and

$n =$ the next term to be found.

Then $(f+n)^3 = f^3 + 3f^2n + 3fn^2 + n^3$. § 182

The work may be arranged as follows:

$$
\begin{array}{c|c|c|lcr}
 & & & \text{Root } = 2a - b & & \\
 & & & \text{Power} = 8a^3 - 12a^2b + 6ab^2 - b^3 \text{ contains} & & \\
 & & & \quad f^3 = 8a^3 & & f^3 + 3f^2n + 3fn^2 + n^3 \\
\cline{4-4}
3f^2 & 3fn & 3f^2 + 3fn & -12a^2b + 6ab^2 - b^3 \text{ contains} & & \\
 & + n^2 & + n^2 & & & 3f^2n + 3fn^2 + n^3 \\
12a^2 & -6ab & 12a^2 - 6ab & -12a^2b + 6ab^2 - b^3 & = & `` \\
 & + b^2 & + b^2 & & & \\
\hline
\end{array}
$$

Explanation. 1. The cube is arranged according to the powers of a and b for a reason similar to that given in square root.

2. $\because 8a^3 =$ the cube of the first term, the first term is $2a$.

3. Subtracting f^3, the remainder, $-12a^2b + 6ab^2 - b^3$, contains $3f^2n + 3fn^2 + n^3$.

4. Dividing by $3f^2$ (*i.e.*, $12a^2$), n is found to be $-b$.

5. $\because f = 2a$, and $n = -b$, $\therefore 3f^2 + 3fn + n^2 = 12a^2 - 6ab + b^2$.

6. Multiplying by n, $-12a^2b + 6ab^2 - b^3$ must equal $3f^2n + 3fn^2 + n^3$. This together with f^3 completes the cube of $f + n$.

Check. Let $a = b = 1$. Then $1^3 = 8 - 12 + 6 - 1 = 1$.

It would be possible, although it is not usually desirable for beginners, to use detached coefficients in extracting cube root. Thus, in the above case

$$
\begin{array}{lll}
 & 8 - 12 + 6 - 1 & \underline{|2 - 1} \\
 & 8 & \\
\cline{2-2}
12 \quad -6 + 1 & -12 + 6 - 1 & \\
12 - 6 + 1 & -12 + 6 - 1 & 2a - b \\
\cline{1-2}
\end{array}
$$

EXERCISE LXXXII

Extract the cube roots of the following expressions:

1. $x^3 + 3x + \dfrac{3}{x} + \dfrac{1}{x^3}$.

2. $8x^9 + 12x^6 + 6x^3 + 1$.

3. $27x^3 + 27x^2 + 9x + 1$.

4. $8x^{15} + 12x^{10} + 6x^5 + 1$.

5. $27x^3 + 54x^2 + 36x + 8$.

6. $8x^6 + 36x^4 + 54x^2 + 27$.

7. $x^3 - 6x^2y + 12xy^2 - 8y^3$.

8. $\dfrac{a^9}{b^{15}} + \dfrac{3a^8}{b^{14}} - \dfrac{5a^6}{b^{12}} + \dfrac{3a^4}{b^{10}} - \dfrac{a^3}{b^9}$.

9. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

10. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

11. $125y^3 + 150y^2z + 60yz^2 + 8z^3$.

12. $p^6 - 3p^5q + 5p^3q^3 - 3pq^5 - q^6$.

13. $a^3x^3 - 12a^2bx^3 + 48ab^2x^3 - 64b^3x^3$.

14. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

15. $a^6 - 6a^5b + 9a^4b^2 + 4a^3b^3 - 9a^2b^4 - 6ab^5 - b^6$.

16. $a^6 + 3a^5b + 6a^4b^2 + 7a^3b^3 + 6a^2b^4 + 3ab^5 + b^6$.

17. $27 - 27p + 90p^2 - 55p^3 + 90p^4 - 27p^5 + 27p^6$.

18. $8x^6 - 36x^5y + 66x^4y^2 - 63x^3y^3 + 33x^2y^4 - 9xy^5 + y^6$.

19. $a^6 - 2a^5b + \frac{5}{3}a^4b^2 - \frac{20}{27}a^3b^3 + \frac{5}{27}a^2b^4 - \frac{2}{81}ab^5 + \frac{1}{729}b^6$.

20. $a^6 - 12a^5b + 54a^4b^2 - 112a^3b^3 + 108a^2b^4 - 48ab^5 + 8b^6$.

21. $x^3 + 3x^2y - 6x^2 + 3xy^2 - 12xy + 12x + y^3 - 6y^2 + 12y - 8$.

22. $8m^6 - 36m^5n + 102m^4n^2 - 171m^3n^3 + 204m^2n^4 - 144mn^5 + 64n^6$.

197. The cube roots of numbers are found by the same general method.

Required the cube root of 139,798,359.

$$\begin{aligned} \text{Root} &= 5 \quad 1 \quad 9 \\ \text{Power} &= 139{,}798{,}359 \text{ cont's } f^3+3f^2n+3fn^2+n^3. \\ f^3 &= \underline{125{,}000{,}000} \end{aligned}$$

$3f^2$	$3fn$ $+ n^2$	$3f^2 + 3fn$ $+ n^2$	14,798,359 contains $3f^2n + 3fn^2 + n^3$ $f = 500$
750,000	15,100	765,100	$7{,}651{,}000 = 3f^2n + 3fn^2 + n^3$ $n = 10$
780,300	13,851	794,151	7,147,359 contains $3f^2n + 3fn^2 + n^3$ $f = 510$ $7{,}147{,}359 = 3f^2n + 3fn^2 + n^3$ $n = 9$

Explanation. 1. $\because$ the highest order of the power is hundred-millions, the highest order of the root is 100's (why ?) and it is unnecessary to look below millions for the cube of 100's. (Why ?)

2. The greatest cube in the hundred-millions is 125,000,000, the cube of 500. $\therefore$ 500 may be called f.

3. Subtracting, 14,798,359 contains $3f^2n + 3fn^2 + n^3$. (Why ?)

4. This is approximately the product of $3f^2$ and n, and hence if divided by $3f^2$ the quotient is approximately n. $\therefore n = 10$.

5. $\therefore 3fn + n^2 = 15{,}100$, and $3f^2 + 3fn + n^2 = 765{,}100$, and this, multiplied by n, equals $3f^2n + 3fn^2 + n^3$.

6. $\because f^3$ has already been subtracted, after subtracting $3f^2n + 3fn^2 + n^3$ there has been subtracted $(f + n)^3$, or 510^3.

7. Calling 510 the second found part, f, it appears that f^3 has been subtracted. $\therefore$ the remainder contains $3f^2n + 3fn^2 + n^3$.

8. The explanation now repeats itself as in square root.

EXERCISE LXXXIII

Extract the cube roots of the following numbers:

1. 10,077,696.
2. 31,855,013.
3. 125.751501.
4. 367,061.696.
5. 997.002999.
6. 2, to 0.001.
7. 3, to 0.001.
8. 5, to 0.001.
9. 47, to 0.001.
10. 551, to 0.001.
11. 975, to 0.001.
12. 221, to 0.001.

REVIEW EXERCISE LXXXIV

1. Simplify $\dfrac{1}{m+\dfrac{1}{1+\dfrac{1+m}{3-m}}}$.

2. Factor $x^5+x^2-4x^3-4$.

3. Multiply $\dfrac{a^3+ab}{a^2+b^2}$ by $\dfrac{a^3-b^3}{ab(a+b)}$.

4. Divide $a^2+1+\dfrac{1}{a^2}$ by $\dfrac{1}{a}-1+a$.

5. Resolve $x^{16}-1$ into five factors.

6. Extract the cube root of $1-x$ to 5 terms.

7. Solve the equation $x^3+9x^2+8x-60=0$.

8. Divide $ax^2-ab^2+b^2x-x^3$ by $(x+b)(a-x)$.

9. Solve the set of equations $\dfrac{x}{5}+\dfrac{y}{6}=18$, $\dfrac{x}{2}-\dfrac{y}{4}=21$.

10. Solve the equation $(a+x)(b+x)=(c+x)(d+x)$.

11. Extract the cube root of $64-48x+9x^2$ to 3 terms.

12. Expand $(1+3a+3a^2+a^3)^2+(1-3a+3a^2-a^3)^2$.

13. Find all of the positive integral roots of $4x+7y=19$.

14. Find the three roots of the equation $x^3-x^2+1=x$.

15. Find the lowest common multiple of

$$a^2 - 4b^2,\quad (a+2b)^2,\quad (a-2b)^3.$$

16. Find the square root of

$$(x+3)(x+4)(x+5)(x+6)+1.$$

17. Solve the equation

$$7 - 2\{6 - 3[5 - 2(4 - \overline{3+2x})]\} = 1.$$

18. Extract the square root of

$$(x-y)^4 - 2(x^2+y^2)(x-y)^2 + 2(x^4+y^4).$$

19. Find the square root of

$$(2a-b)^2 - 2(2a^2 - 5ab + 2b^2) + (a-2b)^2.$$

20. Extract the cube root of

$$8a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6.$$

21. Prove that

$$[(x-y)^2+(y-z)^2+(z-x)^2]^2 = 2[(x-y)^4+(y-z)^4+(z-x)^4].$$

22. Extract the cube root of

$$a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5$$
$$+ 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9.$$

23. Find the highest common factor of $6a^4 + a^3 - a$ and $4a^3 - 6a^2 - 4a + 3$.

24. Divide the product of $x^2 + x - 2$ and $x^2 + x - 12$ by the sum of $2x^2 + 6x + 1$ and $2 - x(10 + x)$.

25. If $a = -3$, $b = 0$, $c = 1$, $d = -2$, find the numerical value of $a - 2\{b + 3[c - 2a - (a-b)] + 2a - (b + 3c)\}$.

26. The length of a field is twice its breadth; another field, which is 50 yds. longer and 10 yds. broader, contains 6800 sq. yds. more than the former; find the size of each.

CHAPTER XII

THE THEORY OF INDICES

I. THE THREE FUNDAMENTAL LAWS OF EXPONENTS

198. It has already been proved that, when m and n are positive integers,

1. $a^m \cdot a^n = a^{m+n}$. § 60
2. $a^m : a^n = a^{m-n}$. § 72
3. $(a^m)^n = a^{mn}$. § 177

It is now proposed to investigate the meaning of the negative and the fractional exponents; that is, to find what meaning should be attached to symbols like 3^{-2}, $8^{\frac{2}{3}}$, $16^{-\frac{3}{4}}$, a^{-n}, $a^{-\frac{1}{n}}$, ⋯.

We shall then proceed to ascertain whether the three fundamental laws given above are true if m and n are fractional, negative, or both fractional and negative.

The necessity for this is apparent. We know that $a^m \cdot a^n = a^{m+n}$, if m and n are positive integers, because a is taken first m times, and then n times, as a factor, and hence $m+n$ times in all. But we do not yet know that $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}$. Neither do we know that $a^{\frac{1}{m}} : a^{\frac{1}{n}} = a^{\frac{1}{m}-\frac{1}{n}}$, nor that $a^{-m} \cdot a^{-n} = a^{-m-n}$, nor that $a^{-m} \cdot a^{\frac{1}{n}} = a^{-m+\frac{1}{n}}$, etc., for we have not yet found what these expressions mean. It certainly is meaningless to say, "We take a minus m times as a factor."

II. THE MEANING OF THE NEGATIVE INTEGRAL EXPONENT

199. The primitive idea of *power* (§ 8) was a product of equal factors. The primitive idea of *exponent* was the number which showed how many equal factors were taken.

According to this primitive idea the

3d power of a meant aaa, written a^3,
2d " " " aa, " a^2;

but there was no first power of a, because that is not the product of any number of a's, nor was there any zero power, fractional power, or negative power.

But since	a^3	means	aaa,	or $a^4 \div a$,
and	a^2	"	aa,	" $a^3 \div a$,
∴ it is reasonable to *define*	a^1	as	a,	" $a^2 \div a$,
and " " " "	a^0	"	1,	" $a \div a$,
" " " " "	a^{-1}	"	$\frac{1}{a}$,	" $1 \div a$,
" " " " "	a^{-2}	"	$\frac{1}{a^2}$,	" $\frac{1}{a} \div a$,
and, in general, to define	a^{-n}	"	$\frac{1}{a^n}$,	

n being a positive integer.

200. For this reason we *define*

a^1 to mean a,
a^0 " 1,
a^{-n} " $\frac{1}{a^n}$,

n being a positive integer, and a not 0.

III. THE MEANING OF THE FRACTIONAL EXPONENT

201. We have now found the meaning of

1. The positive integral exponent greater than 1, the primitive meaning of exponent;
2. The unit exponent;
3. The zero exponent;
4. The negative integral exponent.

202. It remains to find the meaning which should attach to the **fractional exponent.**

The expression a^4 means $aaaa$,

and if the exponent is half as large,

a^2 or aa is the square root of a^4,

and if the exponent is half as large,

a^1 or a is the square root of a^2.

$\therefore$ if an exponent half as large indicates a square root,

$a^{\frac{1}{2}}$ should mean the square root of a.

Hence, $a^{\frac{1}{2}}$ is defined to mean the square root of a, and, in general, $a^{\frac{1}{n}}$ is defined to mean the nth root of a.

203. The reason for this is also seen from the fact that

$\because$ $a^m \cdot a^m \cdots$ to n factors $= a^{mn}$,

$\therefore$ $a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots$ " " should equal $a^{n\left(\frac{1}{n}\right)}$ or $a^{\frac{n}{n}}$ or a.

$\therefore$ $a^{\frac{1}{n}}$ should be defined to mean the nth root of a.

204. And since $a^{mn} = (a^m)^n$, so $a^{\frac{p}{q}}$ should be defined to be identical with $(a^{\frac{1}{q}})^p$.

Hence, *we define* $a^{\frac{p}{q}}$ *to mean the* p*th power of the* q*th root of* a, *and* $a^{-\frac{p}{q}}$ *to mean the reciprocal of* $a^{\frac{p}{q}}$.

ILLUSTRATIVE PROBLEMS

1. Find the absolute value of $343^{-\frac{2}{3}}$.

$$343^{-\frac{2}{3}} = \frac{1}{(343^{\frac{1}{3}})^2} = \frac{1}{7^2} = \frac{1}{49}.$$

2. Write in integral form, with negative or fractional exponents $1 \div \sqrt[7]{x^6}$, or, as in § 206, $1 \div (\sqrt[7]{x})^6$.

$$1 \div \sqrt[7]{x^6} = 1 \div x^{\frac{6}{7}} = x^{-\frac{6}{7}}.$$

3. Write without negative or fractional exponents $a^{-\frac{1}{5}}$.

$$a^{-\frac{1}{5}} = \frac{1}{a^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{a}}.$$

EXERCISE LXXXVI

Find the absolute value of the expressions in Exs. 1–15.

1. $4^{\frac{1}{2}}$. 2. $9^{\frac{1}{2}}$. 3. $8^{\frac{1}{3}}$. 4. $32^{\frac{1}{5}}$. 5. $81^{\frac{1}{4}}$.

6. $25^{\frac{3}{2}}$. 7. $125^{\frac{1}{3}}$. 8. $32^{\frac{3}{5}}$. 9. $64^{\frac{5}{6}}$. 10. $625^{\frac{3}{4}}$.

11. $16^{-\frac{1}{2}}$. 12. $36^{-\frac{3}{2}}$. 13. $27^{\frac{2}{3}}$. 14. $16^{-\frac{3}{4}}$. 15. $32^{\frac{3}{5}}$.

Write in integral form, with negative or fractional exponents, the expressions in Exs. 16–23.

16. $\sqrt{1 \div a^3}$.

17. $\sqrt[3]{1 \div a^2}$.

18. $\sqrt{1 \div b^3\sqrt{c}}$.

19. $\sqrt[3]{1 \div (a+b)^2}$.

20. $\dfrac{a}{\sqrt[4]{b}} + \dfrac{a+b}{\sqrt[3]{a-b}} - \dfrac{\frac{1}{a}}{\sqrt{a}+\sqrt[3]{a}} - \dfrac{1}{a^2}$.

21. $a\sqrt[3]{b} + b\sqrt[3]{a} + \sqrt[3]{a+b} - \sqrt[n]{a-b}$.

22. $\dfrac{1}{a\sqrt{a}} + \dfrac{1}{b\sqrt{b}} - \dfrac{\sqrt{a^3}+\sqrt{b^3}}{\sqrt{a^3}} + \dfrac{1}{a^3} - \dfrac{1}{b^3}$.

23. $a^2 \div (\sqrt{a} + \sqrt{1 \div a}) \div (a^3 + \sqrt[3]{a^2}) \div \sqrt{a^3}$.

Write the following using the old form of radical sign ($\sqrt{\ }$) and the common fraction:

24. $z^{\frac{1}{6}}$. 25. $x^{\frac{m}{n}}$. 26. $a^{-\frac{1}{5}}$. 27. $x^{\frac{m+1}{2}}$.

28. $a^{\frac{1}{2}}b^{\frac{1}{3}}$. 29. $a^{\frac{2}{3}}b^{\frac{1}{3}}$. 30. $x^{\frac{1}{2}}y^{\frac{1}{3}}$. 31. $a^{-\frac{2}{3}}b^{-\frac{2}{3}}$.

32. $x^{\frac{7}{8}}y^{-\frac{7}{8}}$. 33. $x^{\frac{m+n}{m-n}}y^{\frac{m-n}{m+n}}$. 34. $m^{\frac{2}{3}}n^{\frac{4}{5}}$. 35. $x^{-1}y^{\frac{1}{2}}$.

205. It has been proved that $(abc \cdots)^m = a^m b^m c^m \cdots$. It is equally true that $(abc \cdots)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} c^{\frac{1}{n}} \cdots$.

For let $\qquad x = a^{\frac{1}{n}} b^{\frac{1}{n}}.$

Then $\qquad x^n = (a^{\frac{1}{n}} b^{\frac{1}{n}})^n = ab.$ § 178

$$\therefore\ x = (ab)^{\frac{1}{n}}.$$

$$\therefore\ (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}.\qquad \text{Ax. 1}$$

The same reasoning holds for $(abc \cdots)^{\frac{1}{n}}$.

Similarly $\qquad \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}.$

206. It has been proved that $(a^m)^n = a^{mn} = (a^n)^m$. It is equally true that $(a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$.

For $\qquad (aaa \cdots \text{ to } m \text{ factors})^{\frac{1}{n}} = a^{\frac{1}{n}} a^{\frac{1}{n}} a^{\frac{1}{n}} \cdots \text{ to } m \text{ factors}.$ § 205

I.e., $\qquad (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m.$

But $\qquad (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}.$ Def. § 204

Hence, $a^{\frac{m}{n}}$ may be considered either as the mth power of the nth root of a or as the nth root of the mth power of a.

But this must be understood to apply only to the *absolute values* of the roots.

E.g., $\qquad (4^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \pm 4,$

but $\qquad (4^{\frac{1}{2}})^2 = (\pm 2)^2 = +4.$

207. Fractional exponents are subject to the laws of common fractions, although they are only fractions in *form*. For example,

$$a^{\frac{m}{n}} = a^{\frac{pm}{pn}}.$$

For let $x = a^{\frac{m}{n}}$.

Then $x^n = a^m$, by raising to nth power,

and $\therefore\ x^{pn} = a^{pm}$, " " " pth "

$\therefore\ x = a^{\frac{pm}{pn}}$.

Therefore $a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$. Ax. 1

Hence, *both terms of a fractional exponent can be multiplied or divided by the same number without altering the value of the expression.*

The student should understand clearly that this is true *not because the exponent is a fraction.* The exponent is merely an expression in the *form* of a fraction, and hence a proof like that of § 118 has no application to this case. The laws of fractions apply to fractional exponents only as they are proved to do so.

208. It has been proved that $(a^m)^n = a^{mn}$. It is equally true that

$$(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}} = (a^{\frac{1}{n}})^{\frac{1}{m}}.$$

For let $x = (a^{\frac{1}{m}})^{\frac{1}{n}}$.

Then $x^n = a^{\frac{1}{m}}$,

and $x^{mn} = a$.

Therefore $x = a^{\frac{1}{mn}}$.

Similarly if $x = (a^{\frac{1}{n}})^{\frac{1}{m}}$,

it follows that $x = a^{\frac{1}{mn}}$.

$$\therefore\ x = (a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}.$$

IV. THE LAWS FOR NEGATIVE AND FRACTIONAL EXPONENTS

209. Having now found the meaning of the negative and the fractional exponents, and having proved certain laws concerning them, it remains to prove that the three fundamental laws of exponents,

$$a^m \cdot a^n = a^{m+n},$$
$$a^m : a^n = a^{m-n},$$
$$(a^m)^n = a^{mn},$$

are true, *if* m *and* n *are fractional, negative, or both fractional and negative.*

210. To prove that $a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$ or $a^{\frac{ps+qr}{qs}}$.

We know from § 207 that

$$\begin{aligned} a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} &= a^{\frac{ps}{qs}} \cdot a^{\frac{qr}{qs}} & \\ &= (a^{ps} \cdot a^{qr})^{\frac{1}{qs}} & \text{§ 205} \\ &= (a^{ps+qr})^{\frac{1}{qs}} & \text{§ 60} \\ &= a^{\frac{ps+qr}{qs}} & \text{§ 204} \end{aligned}$$

This shows that a case like $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ can be easily handled by fractional exponents, thus:

$$a^{\frac{2}{3}} \cdot a^{\frac{4}{5}} = a^{\frac{2}{3}+\frac{4}{5}} = a^{\frac{22}{15}}.$$

To see that $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ equals the 15th root of a^{22} is not so easy by the help of the old symbols alone.

211. To prove that $a^{\frac{p}{q}} : a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}$.

The proof is evidently identical with that just given, except that the sign of division replaces that of multiplication in the first member, and the sign of subtraction that of addition in the second member.

212. These laws (§§ 205–211) are of great importance in simplifying algebraic expressions.

ILLUSTRATIVE PROBLEMS

1. Simplify $(4\,a^2b^6)^{\frac{1}{2}}$.

By § 205,
$$(4\,a^2b^6)^{\frac{1}{2}} = 4^{\frac{1}{2}}\,a^{\frac{2}{2}}b^{\frac{6}{2}}$$
$$= 2\,ab^3. \qquad \text{§ 210}$$

2. Simplify $(2\,a^{\frac{1}{2}}b^{\frac{1}{4}})^4$.

By § 178,
$$(2\,a^{\frac{1}{2}}b^{\frac{1}{4}})^4 = 2^4\,a^{\frac{4}{2}}b^{\frac{4}{4}}$$
$$= 16\,a^2b. \qquad \text{§ 207}$$

3. Simplify $\sqrt[3]{a^6b^9}$.

By definition of fractional exponent,
$$\sqrt[3]{a^6b^9} = (a^6b^9)^{\frac{1}{3}}$$
$$= a^2b^3. \qquad \text{§ 205}$$

The same result may be found by § 191.

4. Simplify $\sqrt{a}\cdot\sqrt[3]{a^2}$.

$$\sqrt{a}\cdot\sqrt[3]{a^2} = a^{\frac{1}{2}}\cdot a^{\frac{2}{3}}$$
$$= a^{\frac{1}{2}+\frac{2}{3}}$$
$$= a^{\frac{7}{6}} = a\cdot a^{\frac{1}{6}} \text{ or } a\sqrt[6]{a}.$$

5. Simplify $\dfrac{\sqrt{a^3}}{\sqrt[3]{a^2}}$.

$$\frac{\sqrt{a^3}}{\sqrt[3]{a^2}} = a^{\frac{3}{2}-\frac{2}{3}} = a^{\frac{5}{6}} \text{ or } \sqrt[6]{a^5}.$$

6. Simplify $a^{\frac{2}{3}}b^{\frac{3}{4}}\cdot a^{\frac{1}{6}}b^{\frac{3}{8}}\cdot b^{\frac{1}{2}}\cdot(a^{\frac{1}{12}})^2\cdot b^{\frac{1}{8}}\cdot b^{\frac{1}{4}}$.

By §§ 206, 207, 210, this equals
$$a^{\frac{2}{3}+\frac{1}{6}+\frac{1}{6}}b^{\frac{3}{4}+\frac{3}{8}+\frac{1}{2}+\frac{1}{8}+\frac{1}{4}},$$
which equals ab^2.

EXERCISE LXXXVII

Simplify the following expressions:

1. $x^{\frac{1}{4}} \cdot x^{\frac{1}{5}}$.
2. $x^{\frac{1}{3}} \cdot x^{\frac{1}{2}} \cdot x$.
3. $(x^7y^{21}z^{35})^7$.
4. $(3\,a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{4}}y^{\frac{1}{4}})^4$.
5. $(a^{\frac{1}{6}}b^{\frac{1}{4}}c^{\frac{1}{3}}d)^{12}$.
6. $(81\,a^{12}x^{20}y^{28})^{\frac{1}{4}}$.
7. $\sqrt[3]{a^6b^{12}x^9y^{15}}$.
8. $(w^{20}x^{40}y^{60}z^{80})^{0.1}$.
9. $(2\,a^{\frac{1}{7}}b^{\frac{1}{7}}cd^2)^7$.
10. $(a^{10}b^{20}c^{30}d^{40})^{\frac{1}{10}}$.
11. $(27\,a^9b^{12}c^3)^{\frac{1}{3}}$.
12. $(-\,a^{\frac{1}{3}}b^{\frac{1}{6}}x^{\frac{1}{4}}y)^{12}$.
13. $\sqrt[6]{a^{12}b^{24}c^{30}d^{42}}$.
14. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}}q^{\frac{1}{7}}r)^{140}$.
15. $(p^{\frac{1}{2}}q^{\frac{1}{3}}r^{\frac{1}{4}}s^{\frac{1}{5}})^{60}$.
16. $\sqrt[3]{-\,m^6n^9x^{15}y^{21}}$.
17. $(64\,a^{12}b^{21}c^6)^{\frac{1}{3}}$.
18. $(125\,a^3b^6c^9d^{12})^{\frac{1}{3}}$.
19. $(a^{12}b^{18}c^{30}d^{36})^{\frac{1}{6}}$.
20. $(343\,m^{12}n^{15}p^{18})^{\frac{1}{3}}$.
21. $(32\,a^{20}b^{30}c^{40})^{\frac{1}{5}}$.
22. $\sqrt[5]{-\,32\,a^{10}b^{20}c^{30}d^{40}}$.
23. $\sqrt[3]{x^2yz} : \sqrt{xyz}$.
24. $x^{\frac{1}{a}} \cdot x^{\frac{3}{a}} : x^{\frac{2}{a}} \cdot x^{\frac{a-2}{a}}$.
25. $\sqrt[4]{x^3yz} : \sqrt[5]{x^4y^3z^2}$.
26. $\sqrt[4]{a^3} \cdot \sqrt{a} \cdot \sqrt[3]{a^2}$.
27. $a^{\frac{1}{2}}b^{\frac{1}{3}} \cdot a^{\frac{1}{3}}b^{\frac{1}{2}} \cdot a^{\frac{1}{6}}b^{\frac{1}{6}}$.
28. $\sqrt[5]{a^2bx^2y} : \sqrt[3]{a^2bx^2y}$.
29. $p \cdot p^{\frac{1}{2}} \cdot p^{\frac{1}{3}} \cdot p^{\frac{1}{4}} : p^{\frac{1}{12}}$.
30. $\sqrt[3]{ab} \cdot \sqrt[4]{a^3b} \cdot \sqrt[6]{a^5b^4}$.
31. $a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}} \cdot a^{\frac{2}{3}}b^{\frac{3}{4}}c^{\frac{2}{5}} \cdot a^2b^3c^{\frac{3}{5}}$.
32. $\sqrt[7]{-\,p^2q^3x^5y^6} \cdot \sqrt[6]{p^3q^4x^3y^4} : \sqrt[5]{pq^2x^3y^4}$.
33. $\sqrt[3]{x^2y^2z \cdot x^4y^4z \cdot xyz} : \sqrt[4]{x^3y^3z^3 \cdot x^2y^2z^2}$.

213. To prove that $a^m \cdot a^{-n} = a^{m+(-n)}$, or a^{m-n}.

$$a^m \cdot a^{-n} \text{ means } a^m \cdot \frac{1}{a^n}$$

$$= \frac{a^m}{a^n} \qquad \S\ 130$$

$$= a^{m-n}. \qquad \S\ 72$$

214. In the case of $a^m : a^{-n} = a^{m-(-n)} = a^{m+n}$, the proof is evidently identical with that just given, except that the sign of division replaces that of multiplication, and the sign of subtraction that of addition.

215. To prove that $a^{-m} \cdot a^{-n} = a^{-m+(-n)} = a^{-m-n}$.

By definition $\quad a^{-m} \cdot a^{-n} = \dfrac{1}{a^m} \cdot \dfrac{1}{a^n} \qquad \S\ 200$

$$= \frac{1}{a^m a^n}$$

$$= \frac{1}{a^{m+n}}$$

$$= a^{-m-n}. \qquad \S\ 200$$

216. It has now been shown that

$$a^m \cdot a^n = a^{m+n}$$

and

$$a^m : a^n = a^{m-n},$$

whether m and n are fractional, negative, or both fractional and negative.

ILLUSTRATIVE PROBLEM

Simplify $\dfrac{1}{\sqrt[5]{a^4}} : \dfrac{1}{\sqrt[4]{a^3}}$.

Here we have

$$a^{-\frac{4}{5}} : a^{-\frac{3}{4}} = a^{-\frac{16-15}{20}} = a^{-\frac{1}{20}},$$

or the 20th root of $\frac{1}{a}$, a result not so easily reached by the older notation.

EXERCISE LXXXVIII

Simplify the following expressions:

1. $\dfrac{a}{\sqrt[3]{a}} \cdot \dfrac{\sqrt{a}}{a^{\frac{1}{6}}}$.

2. $\dfrac{1}{\sqrt[3]{a}} : \dfrac{1}{\sqrt[4]{a^3}}$.

3. $\dfrac{1}{\sqrt[5]{x^4}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[15]{x^{13}}}$.

4. $\dfrac{1}{\sqrt[3]{a^2b}} \cdot \dfrac{1}{\sqrt{ab^3}}$.

5. $\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{a^3b}} : \dfrac{\sqrt[6]{a^3b^2}}{\sqrt[12]{a^7b^9}}$.

6. $\dfrac{a^2}{a^{\frac{1}{3}}} \cdot \dfrac{a^{\frac{1}{5}}}{a^{\frac{1}{15}}} \quad \sqrt[5]{a^4}$.

7. $\dfrac{a^2}{\sqrt[3]{a^2}} : \dfrac{\sqrt{a}}{a^3} : \dfrac{\sqrt[3]{a}}{\sqrt{a}}$.

8. $a^{-2} \cdot a^{\frac{2}{3}} \cdot a^3 \cdot \sqrt[3]{a}$.

217. It remains to prove that the law that $(a^m)^n = a^{mn}$ is true if m and n are fractional, negative, or both fractional and negative. Since the proofs are so nearly like those already given, only a single case need be considered.

218. To prove that $(a^m)^{-n} = a^{-mn}$.

By definition $\quad (a^m)^{-n} = \dfrac{1}{(a^m)^n}$, § 200

$= \dfrac{1}{a^{mn}}$, § 177

$= a^{-mn}$. § 200

ILLUSTRATIVE PROBLEMS

1. Simplify $\quad \sqrt[3]{\left(\dfrac{1}{1 \div \sqrt[4]{a^3}}\right)^4}$.

This expression, thus written in the older style, does not strike the eye as simple; but since $1 \div \sqrt[4]{a^3}$ may be written $a^{-\frac{3}{4}}$, the expression reduces to $(a^{\frac{3}{4}})^{\frac{4}{3}}$, which equals a.

2. Simplify $$x\cdot\sqrt[r^2-q^2]{\left(\frac{1}{\sqrt[q]{x^r}}\div\frac{1}{\sqrt[r]{x^q}}\right)^{qr}}.$$

Writing this with fractional and negative exponents, we have

$$x\cdot\left(x^{-\frac{r}{q}+\frac{q}{r}}\right)^{\frac{qr}{r^2-q^2}}=x\cdot x^{-\frac{r^2-q^2}{qr}\cdot\frac{qr}{r^2-q^2}}=x\cdot x^{-1}=x^0=1.$$

To simplify this without the assistance of negative and fractional exponents would be more difficult.

EXERCISE LXXXIX

Simplify the following expressions:

1. $(a^{\frac{3}{4}}b^{\frac{5}{6}})^{\frac{2}{3}}$.

2. $a^{\frac{2}{3}}\cdot a^{\frac{2}{5}}$.

3. $[(-a^3)^2]^5$.

4. $\sqrt[3]{x^7y^6z^5}$.

5. $(x^{-1}+y^{-1})^3$.

6. $\sqrt[p]{a^{-q}b^{2p}}$.

7. $[(x^{\frac{2}{3}}y^{-\frac{5}{9}}z^{\frac{4}{5}})^{-\frac{7}{8}}]^{\frac{1}{4}}$.

8. $\sqrt[m]{a^{2m}b^{3m^2}}$.

9. $(2\,x^{-2}\div y^{-2})^{-3}$.

10. $\sqrt[m+n]{a^{m^2-n^2}}$.

11. $\sqrt[13]{(\sqrt[3]{x^2}\cdot\sqrt{x^4})^{14}}$.

12. $a^{-\frac{p}{q}}b^{-x}\sqrt[3]{cd^2}$.

13. $\{[(x^{-2})^{-2}]^{-2}\}^{-2}$.

14. $x^{-x}\cdot(-x^x)$.

15. $(x^{m-n})^{m+n}\cdot x^{n^2}:x^{m^2}$.

16. $\sqrt[m]{a^{-m}b^{-2m}c^{-m^2}}$.

17. $\sqrt[3]{64\,[(x-y)^{-6}]^{\frac{1}{2}}}$.

18. $5\sqrt[3]{x^2y}\cdot 2\,x^{\frac{3}{4}}y^2$.

19. $\{[(x^2-y^{-2})^{-1}]^3\}^{-2}$.

20. $ax^{-m}y^{-n}\cdot bx^ny^m$.

21. $[(a^{m+n})^{m-n}\cdot(a^{n^3})^{\frac{1}{n}}]^{\frac{1}{m^2}}$.

22. $3\,a^{-\frac{2}{3}}\cdot 4\,a^{-\frac{5}{6}}\cdot 2\,a^{\frac{3}{4}}$.

23. $[(a^{-p})^q]^{-\frac{1}{p}}:[(a^q)^{-r}]^{-\frac{1}{r}}$.

24. $3\,a^{-2}b^{-3}c^4\cdot 4\,a^{-3}b^5c^{-4}$.

25. $-a^{-4}b^4c^5d^{-5}\cdot -a^5b^{-5}c^4d^{-4}$.

V. PROBLEMS INVOLVING FRACTIONAL AND NEGATIVE EXPONENTS

219. It has now been proved that we can operate with expressions involving negative or fractional exponents just as if these exponents were positive integers. Exercises involving such exponents will be given on pp. 224, 225.

The student should see the distinct advantage in using the fractional exponent instead of the old form of radical sign, except in cases like the expression of a single root, *and in using the negative exponent,* except in cases like the expression of a simple fraction. This has been shown on pp. 220, 221, but it is worth while to consider the matter further, that the student may become entirely familiar with the use of the modern symbols.

E.g., while it is easier to write $\sqrt{a}$ than $a^{\frac{1}{2}}$, and $\frac{1}{a}$ than a^{-1}, because we are more accustomed to the forms $\sqrt{a}$ and $\frac{1}{a}$, it is much easier to see that

$$(x^{-\frac{2}{3}})^{-\frac{3}{4}} = x^{\frac{1}{2}},$$

than to see that the equivalent expression

$$\frac{1}{\sqrt[4]{(1 \div \sqrt[3]{x^2})^3}} = \sqrt{x}.$$

To take another example, it is doubtful if students would readily grasp the significance of the form $a^3 + 2a^2\sqrt[6]{a} + a\sqrt[3]{a}$; but when written $a^{\frac{6}{2}} + 2a^{\frac{13}{6}} + a^{\frac{4}{3}}$, it is seen to be the square of $a^{\frac{3}{2}} + a^{\frac{2}{3}}$.

In the case of polynomials the value of the negative and fractional exponents is also quite as evident as in that of monomials.

E.g., the eye more readily takes in the operation suggested by the symbols

$$(x^{\frac{5}{6}} - y^{\frac{1}{6}})(2x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}),$$

than the same operation expressed by the symbols

$$(\sqrt[6]{x^5} - \sqrt[6]{y})\left(\frac{2}{\sqrt[3]{x^2}} + \frac{3}{\sqrt[3]{x}\sqrt[3]{y}} + \frac{1}{\sqrt[3]{y^2}}\right).$$

ILLUSTRATIVE PROBLEMS

1. Remove the parentheses from $(x^{-1} \div y^{-1})^{-2}$, expressing the result with positive exponents.

$$(x^{-1} \div y^{-1})^{-2} = x^2 \div y^2. \qquad \S\ 218$$

2. Multiply $x^{-2} + x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Since we can multiply as if the exponents were positive, we have the following:

$$\begin{array}{lllll}
x^{-2} + x^{-1} + 1 & & & & \\
x^{-2} - x^{-1} + 1 & & & & \\
\hline
x^{-4} & + x^{-3} & + x^{-2} & & \\
 & - x^{-3} & - x^{-2} & - x^{-1} & \\
 & & + x^{-2} & + x^{-1} & + 1 \\
\hline
x^{-4} & & + x^{-2} & & + 1
\end{array}$$

Detached coefficients may be used in practice.

3. Divide $x^{-3} + 3x^{-2} + 3x^{-1} + 1$ by $x^{-1} + 1$.

Since we can divide as if the exponents were positive, we have the following:

$$\text{Quotient} = x^{-2} + 2x^{-1} + 1$$

$$\begin{array}{r|l}
x^{-1} + 1 & x^{-3} + 3x^{-2} + 3x^{-1} + 1 \\
 & \underline{x^{-3} + x^{-2}\phantom{+ 3x^{-1} + 1}} \\
 & \phantom{x^{-3} +\ } 2x^{-2} + 3x^{-1} \\
 & \phantom{x^{-3} +\ } \underline{2x^{-2} + 2x^{-1}} \\
 & \phantom{x^{-3} + 2x^{-2} +\ } x^{-1} + 1 \\
 & \phantom{x^{-3} + 2x^{-2} +\ } \underline{x^{-1} + 1}
\end{array}$$

Detached coefficients may be used in practice.

It is evident that we may check the work by arbitrary values as in the case of positive integral exponents.

Thus Ex. 3, let $x = 1$; then

$$(1 + 3 + 3 + 1) \div (1 + 1) = 1 + 2 + 1,$$

$$8 \div 2 = 4.$$

EXERCISE XC

Perform the operations indicated in the following exercises:

1. $\dfrac{a^{\frac{4}{5}}b^{\frac{3}{4}}}{c^{\frac{2}{3}}} : \dfrac{a^{\frac{2}{5}}b^{\frac{2}{3}}}{c^{\frac{5}{6}}}$.

2. $(x^{\frac{m}{n}} + x^{\frac{n}{m}})^2$.

3. $4\,ab^{\frac{2}{3}}c^{\frac{1}{4}} : 2\,b^{\frac{4}{5}}c^{\frac{7}{8}}$.

4. $(x^{-\frac{1}{n}} - 1)^2$.

5. $\sqrt{a^{-2} + 2 + a^2}$.

6. $(a^{\frac{1}{3}} + b^{\frac{2}{3}})^2$.

7. $3\,a^{\frac{1}{3}}b^{\frac{1}{4}} : 1.5\,a^{\frac{1}{6}}b^{\frac{1}{8}}$.

8. $(a^{\frac{1}{3}} - b^{-\frac{1}{3}})^3$.

9. $a^{\frac{1}{2}}b^{\frac{5}{7}} : a^{\frac{1}{4}}b^{\frac{4}{7}} \cdot a^{\frac{3}{4}}b^{\frac{6}{7}}$.

10. $(a^{-b} - b^{-a})^3$.

11. $(x + y)(x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

12. $(x^{\frac{2}{3}} + y^{-\frac{1}{3}})^2$.

13. $\sqrt{9\,x^{\frac{4}{7}} + 6\,x^{\frac{2}{7}} + 1}$.

14. $(a^{\frac{1}{4}}b^{\frac{1}{2}}c + 1)^2$.

15. $\sqrt{4\,x^{\frac{2}{5}} + 4\,x^{\frac{1}{5}} + 1}$.

16. $(a^{-1} + a^{-2})^{-2}$.

17. $a^{\frac{2}{3}}b^{\frac{3}{7}} \cdot a^{\frac{5}{6}}b^{\frac{1}{14}} : a^{\frac{2}{3}}b^{\frac{1}{7}}$.

18. $\dfrac{a^{\frac{2}{3}}b}{ab^{\frac{2}{3}}} \cdot a^{\frac{1}{3}} \cdot \dfrac{ab^{\frac{2}{3}}}{b^{\frac{1}{3}}}$.

19. $(a^{-1} + a^{-2} + a^{-3})^2$.

20. $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)^2$.

21. $\sqrt{a^{\frac{2}{3}} + 2\,a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$.

22. $2\,a^{\frac{1}{2}}b^{\frac{1}{4}} \cdot 3\,a^{\frac{1}{4}}b^{\frac{1}{2}}$.

23. $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} - y^{\frac{2}{3}})$.

24. $\sqrt{x^{-4} + x^4 - 2}$.

25. $(a^2 - a^{-2})(a^2 + a^{-2})$.

26. $3\,a^{\frac{2}{3}}x^{\frac{1}{4}} \cdot 6\,a^{\frac{1}{6}}x^{\frac{1}{2}}$.

27. $a^{\frac{2}{3}}b^{\frac{1}{3}}c \cdot a^{\frac{1}{6}}b^{\frac{1}{6}}c^{\frac{5}{6}} \cdot a^{\frac{5}{6}}b^{\frac{1}{3}}c^{\frac{1}{3}}$

28. $(a^2b^{-2}c^{\frac{1}{2}}d^{-\frac{1}{2}} + 1)^2$.

29. $\sqrt[3]{x^{\frac{3}{5}} + 3\,x^{\frac{2}{5}} + 3\,x^{\frac{1}{5}} + 1}$.

30. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} + y^{\frac{2}{3}})$.

31. $(x^{-\frac{1}{2}} + y^{-\frac{1}{2}})(x^{-\frac{1}{2}} - y^{-\frac{1}{2}})$.

32. $(x^{\frac{1}{3}}y^{\frac{1}{4}}z^{\frac{1}{5}} : x^{-\frac{1}{3}}y^{-\frac{1}{4}}z^{-\frac{1}{5}})^{30}$.

33. $\left(m^{-2} - \dfrac{1}{n^2}\right) : \left(\dfrac{1}{m} - n^{-1}\right)$.

34. $(a^{-3} + a^3)(a^3 - a^{-3})$.

35. $(a^2x^2 - y^{-3})(a^2x^2 + y^{-3})$.

36. $(x^{-a}y^{-b}z^{-c} : x^ay^bz^c)^{-abc}$.

37. $(2^{-2}x^{-4} - x^{-2}y^{-3} + y^{-6})^{\frac{1}{2}}$.

38. $[(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}})]^2$.

39. $ax^{\frac{2}{3}} \cdot a^{\frac{1}{3}}x^{\frac{1}{3}} \cdot a^{\frac{1}{6}}x^{\frac{1}{6}} \cdot a^{\frac{5}{6}} : x^{\frac{1}{6}}$.

40. $(x^{\frac{m}{n}} + y^{\frac{n}{m}})^2 \cdot (x^{\frac{m}{n}} - y^{\frac{n}{m}})^2$.

41. $(x^{-m}-1)^3 \cdot (x^{-m}+1)^3$.

42. $(a-b):(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})$.

43. $(a^{-2}\sqrt{x}-2\sqrt[4]{x}:a+1)^{\frac{1}{2}}$.

44. $\sqrt[3]{8\,a^{\frac{3}{7}}+12\,a^{\frac{2}{7}}+6\,a^{\frac{1}{7}}+1}$.

45. $(a^{\frac{1}{2}}b^{\frac{2}{3}}+a^{\frac{2}{3}}b^{\frac{1}{2}})(a^{\frac{1}{2}}b^{\frac{2}{3}}-a^{\frac{2}{3}}b^{\frac{1}{2}})$.

46. $(a^{\frac{7}{8}}b^{\frac{5}{6}}-a^{\frac{3}{4}}b^{\frac{5}{7}}+4\,a^{\frac{3}{8}}b^{\frac{2}{5}}):a^{\frac{2}{3}}b^{\frac{4}{5}}$.

47. $2\,a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}\cdot 3\,a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}}\cdot 4\,a^{\frac{1}{6}}b^{\frac{5}{12}}c^{\frac{11}{20}}$.

48. $(a^{-2}b^{-4}+c^{-6}+2\,a^{-1}b^{-2}\div c^{3})^{\frac{1}{2}}$.

49. $(a^{-10}-a^{-5}+1):(a^{-2}-a^{-1}+1)$.

50. $(4\,x^{-4}+11\,x^{-2}-45):(2\,x^{-1}-3)$.

51. $(x^{-1}+y^{-1}+z^{-1})(x^{-1}+y^{-1}-z^{-1})$.

52. $(3^{-2}x^{-4}-2^{-4}y^{-2})(3^{-2}x^{-4}+2^{-4}y^{-2})$.

53. $(a^{\frac{1}{2}}+b^{\frac{1}{2}}+a^{-\frac{1}{2}}b)(ab^{-\frac{1}{2}}-a^{\frac{1}{2}}+b^{\frac{1}{2}})$.

54. $(a^{-4}x^{-4}-b^{-2}y^{-2}):(a^{-2}x^{-2}+b^{-1}y^{-1})$.

55. $(4\,x^{\frac{4}{3}}y^{\frac{5}{3}}-9\,x^{\frac{8}{5}}y^{\frac{7}{4}}):(2\,x^{\frac{2}{3}}y^{\frac{5}{6}}+3\,x^{\frac{4}{5}}y^{\frac{7}{8}})$.

56. $(x^{-3}+2\,x^{-2}y^{-1}-3\,y^{-3}):(x^{-1}-y^{-1})$.

57. $(x^{-2}-3\,x^{-\frac{4}{3}}\sqrt[3]{y^{-2}}+3\,x^{-\frac{2}{3}}y^{-\frac{4}{3}}-y^{-2})^{\frac{1}{3}}$.

58. $\sqrt[3]{8^{-1}x^3-3\cdot 2^{-1}x^2y^{\frac{1}{2}}+6\,xy-8\,y\sqrt{y}}$.

59. $3\,a^{-\frac{3}{4}}:5\,a^{-\frac{5}{6}},\ x^2:\{[x^{-\frac{1}{2}}y^{-\frac{1}{3}}(x^2y^2)^{\frac{2}{3}}]^{-\frac{1}{2}}\}^{-6}$.

60. $(x^2+2\,xy+y^2)(x^{-2}-2\,x^{-1}y^{-1}+y^{-2})$.

61. $(16\,x^{-3}+6\,x^{-2}+5\,x^{-1}-6):(2\,x^{-1}-1)$.

62. $(a^{\frac{2}{3}}-a^{\frac{1}{3}}+1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}})(a^{\frac{1}{3}}+1+a^{-\frac{1}{3}})$.

63. $\sqrt[3]{4\,x^{-1}y^2z^{\frac{1}{2}}}:[1:\sqrt{12\,x^3y^{-\frac{2}{3}}z^2}\cdot\sqrt[12]{108\,x^{-3}y^2z^{-4}}]$.

64. $(x^{-3}+3\,x^{-2}y+3\,x^{-1}y^2+y^3)(x^{-2}+2\,x^{-1}y+y^2)$.

65. $(x^{-5}-2x^{-4}-4x^{-3}+19x^{-2}-31x^{-1}+15):(x^{-3}-7\,x^{-1}+5)$.

VI. IRRATIONAL NUMBERS. SURDS

220. Rational and irrational algebraic expressions have already been defined (§ 78). But in algebra it is often necessary to use numbers which are irrational.

221. A **rational number** is a number expressible as the quotient of two integers.

E.g., $3 = \frac{3}{1}$, $0.666 \cdots = \frac{2}{3}$, $\frac{5}{7}$.

222. An **irrational number** is a number which is not rational.

E.g., $2^{\frac{1}{2}}$ or $\sqrt{2}$, $(1 + 2^{\frac{1}{2}})^{\frac{1}{3}}$ or $\sqrt[3]{1+\sqrt{2}}$, $\sqrt{-1}$.

223. Irrational numbers which are not even roots of negative numbers are often called *surds*.

224. Surds are classified as follows:

1. According to the root index, as

quadratic, or of the second order,		as	$\sqrt{5}$,
cubic,	" " third "	"	$\sqrt[3]{7}$,
quartic, or biquadratic,		"	$\sqrt[4]{x}$,
quintic,		"	$\sqrt[5]{5}$,
sextic,		"	$\sqrt[6]{a}$, etc.

2. **Similar** or **dissimilar** (if they have a single term), according as the surd factors are or are not the same.

E.g., $2\sqrt{3}$, $4\sqrt{3}$, $-7\sqrt{3}$ are similar surds.
$2\sqrt{3}$, $3\sqrt{2}$ are dissimilar surds.

3. **Pure** or **mixed** (if they have a single term), according as they do not or do contain either rational factors or dissimilar surd factors.

E. g., $\sqrt{3}$ is a pure surd, but $2\sqrt{3}$ and $\sqrt{5} \cdot \sqrt[3]{3}$ are mixed surds.

4. According to the number of terms in the expression when simplified, as

monomial surds, as $\sqrt{2}$, $3\sqrt[3]{2}$,

binomial " " $\sqrt{2}+\sqrt[3]{5}$, $5+\sqrt{2}$,

trinomial " " $2+\sqrt{3}+\sqrt[4]{7}$,

and, in general, **polynomial** surds.

5. According to simplicity. A surd is said to be in its *simplest form* when the index is as small as possible, and when the expression under the radical sign is integral and contains as a factor no power of the same degree as the index.

E.g., $\sqrt{9}$, $\sqrt[4]{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{a^3x}$, are not in the simplest form. For

$$\sqrt{9}=3,$$
$$\sqrt[4]{4}=\sqrt{\sqrt{4}}=\sqrt{2},$$
$$\sqrt{\tfrac{1}{2}}=\sqrt{\tfrac{2}{4}}=\sqrt{\tfrac{1}{4}\cdot 2}=\tfrac{1}{2}\sqrt{2}.$$
$$\sqrt[3]{a^3x}=a\sqrt[3]{x}.$$

The fractional exponent is, in general, more convenient in all operations involving surds. The two forms of the radical symbol are used here in order that both may be familiar.

225. Convention as to signs. When we consider an expression like $\sqrt{4}+\sqrt{9}$ we see that it reduces to $(\pm 2)+(\pm 3)$, and hence to

$$+2+3=5,$$
$$+2-3=-1,$$
$$-2+3=1,$$
$$-2-3=-5.$$

But for simplicity it is agreed among mathematicians that in expressions of this kind *only the absolute values of the roots shall be considered*, unless the contrary is stated.

Hence, $\sqrt{4}+\sqrt{9}=2+3=5$.

EXERCISE XCI

Classify expressions 1–8 according to the index of the root.

1. $\sqrt[4]{5}$. **2.** $\sqrt[5]{7}$. **3.** $a^{\frac{1}{3}}$. **4.** $x^{\frac{1}{4}}$.

5. $\sqrt[6]{7}$. **6.** $\sqrt{129}$. **7.** $\sqrt[3]{205}$. **8.** $323^{\frac{1}{5}}$.

Classify expressions 9–12 as similar or dissimilar.

9. $2\sqrt{2},\ 5\sqrt{2},\ 8\cdot 2^{\frac{1}{2}}$. **10.** $2\sqrt[3]{5},\ -\sqrt[3]{5},\ \frac{1}{2}\sqrt[3]{5}$.

11. $3\sqrt{2},\ 2\sqrt{3},\ 6\cdot 2^{\frac{1}{2}}$. **12.** $4\sqrt[4]{5},\ 3\sqrt[3]{4},\ 2\sqrt[2]{3}$.

Classify expressions 13–20 as pure or mixed.

13. $\sqrt{47}$. **14.** $3\sqrt[3]{5}$. **15.** $ab^{\frac{1}{2}}$. **16.** $\sqrt{2}\cdot\sqrt[3]{3}$.

17. $\sqrt[3]{10}$. **18.** $2\sqrt[5]{5}$. **19.** $31\sqrt[6]{4}$. **20.** $\sqrt{6}$.

Classify expressions 21–26 according to the number of terms.

21. $a^{\frac{1}{2}}b^{\frac{1}{2}}$. **22.** $\sqrt{2}+\sqrt[3]{5}$. **23.** $2+\sqrt{3}+\sqrt[3]{4}$.

24. $\sqrt{a}+\sqrt{b}$. **25.** $\sqrt[3]{2}+\sqrt[3]{3}+\sqrt[3]{4}$. **26.** $\sqrt[4]{x^2y^3z}$.

Find the value of the expressions in Exs. 27–36.

27. $\sqrt{4}+\sqrt{9}+\sqrt{16}$. **28.** $\sqrt[3]{27}+\sqrt[4]{16}+\sqrt[5]{32}$.

29. $\sqrt[3]{125}+\sqrt[3]{64}+\sqrt[3]{216}$. **30.** $\sqrt{144}+\sqrt{81}+\sqrt{625}$.

31. $\sqrt[3]{8}+\sqrt{25}+\sqrt[4]{16}+\sqrt[5]{32}$.

32. $\sqrt[5]{32}-\sqrt[4]{16}+\sqrt[3]{125}-\sqrt[4]{625}$.

33. $\sqrt[5]{a^5}-\sqrt{121}+\sqrt[3]{1331}-\sqrt[7]{a^7}$.

34. $\sqrt[3]{1728}-\sqrt{144}+\sqrt{169}-13$.

35. $\sqrt[6]{64}+\sqrt[5]{243}+\sqrt[4]{2401}+\sqrt[3]{512}$.

36. $\sqrt{4}+\sqrt[3]{8}+\sqrt[4]{16}+\sqrt[5]{32}+\sqrt[6]{64}$.

226. Reduction of surds. It has been shown (§ 203) that $a = (a^n)^{\frac{1}{n}}$. Hence, it follows that *a number can be reduced to the form of a surd of any order.*

ILLUSTRATIVE PROBLEMS

1. To reduce 2 to the *form* of a surd of the 3d order.

$$2 = (2^3)^{\frac{1}{3}} = \sqrt[3]{8}.$$

$\sqrt[3]{8}$ is not a surd, but it is in the *form* of a surd.

2. To reduce $\sqrt{2}$ to the *form* of a surd of the 5th order.

$$2^{\frac{1}{2}} = (2^{\frac{5}{2}})^{\frac{1}{5}} = \sqrt[5]{32^{\frac{1}{2}}}.$$

3. To reduce $\sqrt[5]{4}$ to the form of a surd of the 10th order.

$$4^{\frac{1}{5}} = 4^{\frac{2}{10}} = \sqrt[10]{16}.$$

Since $a\sqrt[n]{b} = \sqrt[n]{a^n b}$, it follows that *mixed surds can always be reduced to pure surds.*

227. Since it is desirable to have the number under the radical sign as small an integer as possible, it is often necessary *to reduce surds to their simplest forms* (§ 224, 5).

E.g.,

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4}\cdot 2} = \tfrac{1}{2}\sqrt{2}.$$

$$\sqrt[3]{135} = \sqrt[3]{3^3\cdot 5} = 3\sqrt[3]{5}.$$

$$\sqrt{\frac{5}{18}} = \sqrt{\frac{5}{3^2\cdot 2}} = \sqrt{\frac{10}{3^2\cdot 2^2}} = \frac{1}{6}\sqrt{10}.$$

Hence, in the case of fractions under the radical sign *we multiply both terms by the smallest number which will make the denominator the required power, then extract the indicated root of the denominator, and reduce the remaining surd as much as possible.*

228. Since in multiplying surds it is desirable to have them of the same order, it is often necessary *to reduce several surds to equivalent surds of the same order*, the order always being as low as possible.

E.g., $\sqrt{2}\cdot\sqrt[3]{3} = 2^{\frac{1}{2}}\cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}}\cdot 3^{\frac{2}{6}} = (2^3\cdot 3^2)^{\frac{1}{6}} = \sqrt[6]{8\cdot 9} = \sqrt[6]{72}.$

ILLUSTRATIVE PROBLEMS

1. Reduce $3\sqrt[3]{5}$ to a pure surd.

$$3\sqrt[3]{5} = \sqrt[3]{3^3\cdot 5} = \sqrt[3]{135}. \qquad \S\ 226$$

2. Reduce $\sqrt{\frac{12}{13}}$ to its simplest form.

$$\sqrt{\frac{12}{13}} = \sqrt{\frac{12\cdot 13}{13^2}} = \sqrt{\frac{4\cdot 3\cdot 13}{13^2}} = \frac{2}{13}\sqrt{39}. \qquad \S\ 227$$

3. Reduce $\sqrt[4]{6}$ and $\sqrt[3]{2}$ to equivalent surds of the same order.

$6^{\frac{1}{4}}$ and $2^{\frac{1}{3}}$ can evidently be reduced to the order indicated by $4\cdot 3$.

$$6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}. \qquad \S\ 207$$

$$2^{\frac{1}{3}} = 2^{\frac{4}{12}} = \sqrt[12]{2^4} = \sqrt[12]{16}.$$

4. Reduce $\sqrt{a^2x^2 + b^2x^2}$ to its simplest form.

$$\sqrt{a^2x^2 + b^2x^2} = \sqrt{x^2(a^2 + b^2)} = x\sqrt{a^2 + b^2}.$$

EXERCISE XCII

Reduce the numbers in Exs. 1–10 to the *form* of surds of the orders indicated.

1. 5, 3d order.
2. 2, 6th order.
3. $\frac{1}{8}$, 4th "
4. 10, 5th "
5. 11, 2d "
6. 12, 3d "
7. -2, 2d "
8. -5, 3d "
9. 3, 5th "
10. -2, 6th "

Reduce the expressions in Exs. 11–28 to pure surds.

11. $2\sqrt{3}$. 12. $ab^{\frac{1}{2}}c$. 13. $2\sqrt[5]{2}$.

14. $5 \cdot 2^{\frac{1}{3}}$. 15. $3\sqrt{2}$. 16. $4\sqrt{a}$.

17. $ab\sqrt{cd}$. 18. $a^{\frac{1}{2}}b^{\frac{1}{3}}c$. 19. $a\sqrt{2a^3}$.

20. $3\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5}$. 21. $a^2b\sqrt[3]{ab}$. 22. $x\sqrt{x+y}$.

23. $(x+y)\sqrt{x-y}$. 24. $(x^2+y^2)\sqrt{x+y}$.

25. $(x-y)\sqrt{\dfrac{x-y}{x+y}}$. 26. $(a-b)\sqrt{\dfrac{1}{a^2-b^2}}$.

27. $(a^2+b^2)\sqrt{\dfrac{1}{a^2+b^2}}$. 28. $(x^2-2)\sqrt{\dfrac{1}{x^2-2}}$.

Reduce the expressions in Exs. 29–38 to the *form* of surds of the orders indicated.

29. $\sqrt[3]{abc^2}$, 9th order. 30. $\sqrt[7]{a^6}$, 14th order.

31. $\sqrt[3]{5}$, 30th " 32. $3^{\frac{1}{5}}$, 15th "

33. $5^{\frac{1}{4}}$, 20th " 34. $10^{\frac{1}{3}}$, 15th "

35. $\sqrt[4]{4}$, 8th " 36. $\sqrt[15]{5}$, 60th "

37. $\sqrt{2}$, 4th " 38. $\sqrt[3]{2}$, 6th "

Reduce the expressions in Exs. 39–46 to equivalent surds of the same order, the order being as low as possible in each case:

39. $\sqrt{a}$, $\sqrt[3]{b}$. 40. $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[7]{2}$.

41. $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{4}}$. 42. $\sqrt[3]{4}$, $\sqrt[4]{3}$, $\sqrt[12]{5}$.

43. $a^{\frac{1}{2}}b^{\frac{2}{3}}$, $a^{\frac{2}{3}}b^{\frac{1}{2}}$. 44. $\sqrt[5]{2}$, $\sqrt[2]{5}$, $\sqrt[20]{3}$.

45. $7^{\frac{1}{2}}$, $9^{\frac{1}{6}}$, $11^{\frac{1}{5}}$. 46. 2, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$.

Reduce the following expressions to their simplest forms:

47. $\sqrt{25}$. **48.** $\sqrt{32}$. **49.** $\sqrt{75}$. **50.** $\sqrt{162}$.

51. $\sqrt{180}$. **52.** $\sqrt{243}$. **53.** $\sqrt{175}$. **54.** $\sqrt{144}$.

55. $\sqrt{a^2b}$. **56.** $\sqrt{a^2b^3}$. **57.** $\sqrt{4\,ab^3}$. **58.** $\sqrt{81\,x^5y^7}$.

59. $2\sqrt{4\,a^2b^3c^4}$. **60.** $3\sqrt{9\,x^3y^4z^2}$. **61.** $5\sqrt{2\,x^7y^{10}z}$.

62. $4\sqrt{81\,x^{10}y^{12}z^{14}}$. **63.** $\sqrt{3\,a^3+a^2}$. **64.** $\sqrt{x^4y+x^3y^2}$.

65. $\sqrt[3]{8\,a^2b^3c^4}$. **66.** $\sqrt[3]{40\,x^2y^3z^9}$.

67. $\sqrt[3]{54\,a^4b^4c^4}$. **68.** $\sqrt[3]{27\,a^5b^9c^{12}}$.

69. $\sqrt[3]{x^4+ax^5}$. **70.** $(x^7+x^6y)^{\frac{1}{3}}$.

71. $\sqrt[3]{m^6x+m^7}$. **72.** $\sqrt[3]{(a+b)^6y^9}$.

73. $a\sqrt[3]{a^4+a^6x}$. **74.** $\sqrt[3]{a^5x^5+a^7x^7}$.

75. $(x^4+x^5y+x^6y^2)^{\frac{1}{2}}$. **76.** $[(a+b)(a^2-b^2)]^{\frac{1}{2}}$.

77. $\sqrt{a^3-2\,a^2b+ab^2}$. **78.** $\sqrt{ab^2c^3(a+b+c)^3}$.

79. $\sqrt[3]{(a^2+ab)(a^4+2\,a^3b+a^2b^2)}$. **80.** $\sqrt[3]{a^4+3\,a^3b+3\,a^2b^2+ab^3}$.

81. $\sqrt{\frac{1}{3}}$. **82.** $\sqrt{\frac{2}{5}}$. **83.** $\sqrt[3]{\frac{1}{2}}$.

84. $\sqrt[3]{\frac{2}{a}}$. **85.** $\sqrt[4]{\frac{1}{8}}$. **86.** $\sqrt{\frac{a^3}{b}}$.

87. $\sqrt[5]{\frac{1}{16}}$. **88.** $\sqrt{\frac{2\,x}{9\,y}}$. **89.** $\sqrt[3]{\frac{a^2b}{ab^2}}$.

90. $\sqrt{\frac{2\,a}{7\,b}}$. **91.** $\sqrt{\frac{abc}{def}}$. **92.** $\sqrt[3]{\frac{4\,a}{9\,b^2}}$.

93. $\sqrt[3]{\frac{abc^2}{xyz^2}}$. **94.** $\sqrt[5]{\frac{a}{81\,b^3}}$. **95.** $\sqrt[3]{\frac{1}{3\,ab^2}}$.

96. $\sqrt[4]{\frac{3\,a^5x^3}{4\,y^3z}}$. **97.** $\sqrt[4]{\frac{2\,ab^2}{27\,a^3b}}$. **98.** $\sqrt[5]{\frac{x^2y^3z}{27\,a^3b^4}}$.

229. Addition and subtraction of surds. Irrational expressions may evidently be added and subtracted the same as rational expressions, by taking advantage of some convenient unit.

ILLUSTRATIVE PROBLEMS

1. Required the sum of $\sqrt{24}$, $\sqrt{54}$, and $-\sqrt{96}$. Here we have, each surd being reduced to its simplest form,

$$\begin{aligned} \sqrt{24} &= \sqrt{4\cdot 6} = 2\sqrt{6} \\ \sqrt{54} &= \sqrt{9\cdot 6} = 3\sqrt{6} \\ -\sqrt{96} &= -\sqrt{16\cdot 6} = \underline{-4\sqrt{6}} \end{aligned}$$

Hence, the sum is $\sqrt{6}$

2. Required the sum of $\sqrt{8}$, $\sqrt{27}$, $-2\sqrt{2}$, and $\sqrt{48}$. Here we have $2\sqrt{2}+3\sqrt{3}-2\sqrt{2}+4\sqrt{3}=7\sqrt{3}$.

3. Add the following:

$$\begin{array}{lr|c} & & \textit{Check.} \\ a\sqrt{x}+ & b\sqrt[3]{x}-c\sqrt[n]{z} & 1 \\ - & c\sqrt[3]{x}+c\sqrt[n]{z} & 0 \\ a\sqrt{x}+ & b\sqrt[3]{x}+c\sqrt[n]{z} & 3 \\ \hline 2a\sqrt{x}+ & (2b-c)\sqrt[3]{x}+c\sqrt[n]{z} & 4 \end{array}$$

In general, however, the sums of surds can only be indicated as $\sqrt{3}+\sqrt[3]{7}$, $-\sqrt[n]{a}+\sqrt[m]{b}$.

EXERCISE XCIII

Find the sum of the expressions in Exs. 1–12.

1. $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{3}{16}}$.
2. $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{9}}$, $\sqrt{\frac{1}{8}}$.
3. $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$.
4. $\sqrt{24}$, $\sqrt{54}$, $\sqrt{6}$.
5. $\sqrt{\frac{1}{5}}$, $\sqrt{\frac{4}{5}}$, $\frac{1}{5}\sqrt{245}$.
6. $\sqrt[3]{5}$, $\sqrt[3]{40}$, $\sqrt[3]{1080}$.
7. $\sqrt{3}$, $\sqrt{75}$, $\sqrt{108}$.
8. $\sqrt{5}$, $\sqrt{125}$, $\sqrt{500}$.
9. $\sqrt{63}$, $\sqrt{28}$, $\sqrt{175}$.
10. $\sqrt{60}$, $\sqrt{135}$, $\sqrt{240}$.
11. $\sqrt[3]{56}$, $\sqrt[3]{189}$, $\sqrt[3]{875}$.
12. $\sqrt[4]{32}$, $\sqrt[4]{162}$, $\sqrt[4]{512}$.

Simplify the following expressions:

13. $\sqrt[3]{\frac{1}{3}} - \sqrt[6]{3^4} + \frac{1}{3}\sqrt[3]{24}.$

14. $\sqrt[3]{a^2} + 2\sqrt[6]{a^4} + 5\sqrt[9]{a^6}.$

15. $(a^6b)^{\frac{1}{3}} - a^2\sqrt[3]{b} + a^2b^3c.$

16. $\sqrt[4]{32} - \sqrt[4]{162} + \sqrt[4]{1250}.$

17. $\frac{1}{a}\sqrt[3]{a^5} + \frac{1}{2}\sqrt[3]{64\,a^2} - 3\sqrt[9]{a^6}.$

18. $\sqrt{72} + \sqrt{108} - \sqrt{32} + \sqrt{243}.$

19. $\sqrt{a^4b} + \sqrt[4]{a^{12}b^2} - \sqrt[5]{a^{10}b^5}\cdot\sqrt{b}.$

20. $\sqrt[3]{\dfrac{1}{27\,a^2}} + \sqrt[3]{\dfrac{a}{27\,b^3}} + b\sqrt[3]{\dfrac{-1}{27\,a^2b^3}}.$

21. $\sqrt{\dfrac{a-b}{a+b}} + \sqrt{\dfrac{a+b}{a-b}} + \sqrt{a^2-b^2}.$

22. $\sqrt[3]{24} + \sqrt[3]{375} - \sqrt[3]{648} + 10\sqrt[3]{3}.$

23. $\dfrac{1}{3\,a}\sqrt[3]{-81\,a^3} + \sqrt[3]{24\,a} + 4\sqrt[3]{3\,a}.$

24. $\sqrt[3]{\dfrac{1}{a}} - \sqrt[3]{a^2b^{-3}} + (a-b)\sqrt[3]{a^{-1}b^{-3}}.$

25. $a\sqrt[5]{\dfrac{-b^3c^4}{a^3}} + b\sqrt[5]{\dfrac{-a^2c^4}{b^2}} + 6\sqrt[15]{a^6b^9c^{12}}.$

26. $\sqrt{147} + \sqrt{243} - \sqrt{363} + \sqrt{432} - \sqrt{507}.$

27. $\sqrt{3\,a^2 + 3\,b^2 - 6\,ab} + \sqrt{6\,ab + 3\,(a^2+b^2)}.$

28. $\sqrt[3]{1715} + \sqrt[3]{3645} + \sqrt[3]{6655} + \sqrt[3]{8640} - 39\sqrt[3]{5}.$

29. $\dfrac{1}{b}\sqrt[3]{a^4} + b\sqrt[3]{\dfrac{1}{a^2}} - \dfrac{1}{ab}\sqrt[3]{a^7 + 3\,a^5b^2 + 3\,a^3b^4 + ab^6}.$

30. $\sqrt[3]{6\,ab\,(a+b) + 2\,(a^3+b^3)} + \sqrt[3]{-2\,b^3} + \sqrt[3]{-2\,a^3}.$

31. $\sqrt[3]{x^4 + 5\,x^3 + 6\,x^2 - 4\,x - 8} + \sqrt[3]{x^4 - 4\,x^3 + 6\,x^2 - 4\,x + 1}.$

230. Multiplication of surds. In general, products involving irrational numbers must be indicated, as $3\sqrt{2}$, or expressed approximately, as $3\sqrt{2} = 3 \cdot 1.414 \cdots = 4.24 \cdots$. By using fractional exponents, the products may sometimes, however, be simplified.

ILLUSTRATIVE PROBLEMS

1. Multiply $\sqrt{3}$ by $\sqrt{2}$.

Since $$a^{\frac{1}{2}}b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}, \qquad \S\ 205$$

$$\therefore\ 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = 6^{\frac{1}{2}} \text{ or } \sqrt{6}.$$

2. Multiply $\sqrt[3]{3}$ by $\sqrt{2}$.

$$\begin{aligned} 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} &= 2^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} \\ &= (2^3 \cdot 3^2)^{\frac{1}{6}} \\ &= \sqrt[6]{8 \cdot 9} = \sqrt[6]{72}. \end{aligned}$$

3. Multiply $2 + \sqrt{3}$ by $4 - 2\sqrt{3}$.

$$\begin{array}{l} 2 + \sqrt{3} \\ \underline{4 - 2\sqrt{3}} \\ 8 + 4\sqrt{3} \\ \underline{\quad - 4\sqrt{3} - 2\sqrt{3^2}} \\ 8 \qquad\qquad - 6 = 2 \end{array}$$

4. Square $3\sqrt{2} + 2\sqrt[3]{3}$.

$$\begin{aligned} (3\sqrt{2} + 2\sqrt[3]{3})^2 &= (3\sqrt{2})^2 + 2(3\sqrt{2})(2\sqrt[3]{3}) + (2\sqrt[3]{3})^2 \\ &= 18 + 12\sqrt[6]{72} + 4\sqrt[3]{9}. \end{aligned}$$

EXERCISE XCIV

Perform the following multiplications:

1. $(5-\sqrt{3})^2$.
2. $(3+7\sqrt{3})^2$.
3. $(4+2\sqrt[3]{3})^2$.
4. $7\sqrt[3]{7}\cdot 8\sqrt{2}$.
5. $2\sqrt{2}\cdot 5\sqrt{6}$.
6. $2\sqrt[3]{4}\cdot 4\sqrt[3]{2}$.
7. $3\sqrt{5}\cdot 4\sqrt{7}$.
8. $3\sqrt[3]{9}\cdot 5\sqrt[3]{3}$.
9. $\sqrt[5]{16}\cdot 5\sqrt[5]{2}$.
10. $\sqrt[5]{2}\cdot 25\sqrt[3]{4}$.
11. $\sqrt[3]{5}\cdot 14\sqrt{6}$.
12. $3\sqrt[3]{7}\cdot 7\sqrt[4]{3}$.
13. $3\sqrt{\frac{3}{4}}\cdot 2\sqrt[3]{\frac{4}{5}}$.
14. $(3-5\sqrt{3})^2$.
15. $4\sqrt[3]{7}\cdot 5\sqrt[3]{49}$.
16. $7\sqrt{32}\cdot 8\sqrt{2}$.
17. $3\sqrt{3}\cdot 2\sqrt{2}$.
18. $\sqrt{7}\cdot\sqrt[3]{7}\cdot\sqrt[6]{7}$.
19. $\sqrt[3]{a^2b^4}\cdot a\sqrt[3]{a^2b}$.
20. $(2+7^{\frac{2}{3}})(2-7^{\frac{2}{3}})$.
21. $\sqrt{\frac{3}{8}}\cdot\sqrt{\frac{1}{18}}\cdot 2\sqrt[3]{27}$.
22. $(4+\sqrt{7})(4-\sqrt{7})$.
23. $2\sqrt{2}\cdot 3\sqrt{3}\cdot 5\sqrt{6}$.
24. $(2+\sqrt{3})(2-\sqrt{3})$.
25. $\sqrt{2}\cdot\sqrt[3]{3}\cdot\sqrt[3]{2}\cdot\sqrt{3}$.
26. $\sqrt{a}\cdot\sqrt[3]{a}\cdot\sqrt[4]{a}\cdot\sqrt[5]{a}$.
27. $(\sqrt{a-b}+\sqrt{a+b})^2$.
28. $2\sqrt{ab^2c^3d^4}\cdot 3\sqrt[3]{ab^2c^3d^4}$.
29. $(2a+b^{\frac{1}{2}})(3a+2b^{\frac{1}{2}})$.
30. $(2+8\sqrt{3})(4-5\sqrt{3})$.
31. $\sqrt[9]{121}\cdot\sqrt[3]{11}\cdot\sqrt[9]{14641}$.
32. $3\sqrt{2}\cdot 2\sqrt[3]{3}\cdot 4\sqrt[4]{5}\cdot 5\sqrt[6]{6}$.
33. $\sqrt[3]{abx^2y}\cdot\sqrt{abx^2y}\cdot\sqrt[4]{abx^2y}$.
34. $\sqrt[3]{\sqrt{a}+\sqrt{b}}\cdot\sqrt[3]{\sqrt{a}-\sqrt{b}}$.
35. $5\sqrt[4]{ax^2\cdot by^3}\cdot 11\sqrt[3]{a^3x\cdot b^3y}$.
36. $(3-5\sqrt{2})^2-(3+5\sqrt{2})^2$.
37. $(\sqrt{2}+\sqrt{3})(2\sqrt{2}-5\sqrt{3})$.
38. $\frac{1}{3}\sqrt{\frac{20}{45}}\cdot 2\sqrt[3]{\frac{2}{64}}\cdot\frac{1}{3}\sqrt[4]{\frac{567}{7}}\cdot 4\frac{1}{2}$.

231. Division of surds. To divide an irrational number *by a rational number* is equivalent to multiplying by the reciprocal of the rational number, and hence it may be considered as a case of multiplication.

E.g., $\dfrac{a+\sqrt{b}}{c}$ is merely $\dfrac{1}{c}\cdot(a+\sqrt{b})$, or $\dfrac{a}{c}+\dfrac{1}{c}\sqrt{b}$,

and $\dfrac{4+\sqrt{3}}{2}$ is merely $\frac{1}{2}(4+\sqrt{3})$, or $2+\frac{1}{2}\sqrt{3}$.

232. Division by a surd usually reduces, without much difficulty, to division by a rational number, as shown in the following example:

To divide $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}$.

Multiplying both terms of the fraction

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{5}}$$

by $\sqrt{5}$, we have

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{5}(\sqrt{2}+\sqrt{3})}{\sqrt{5}\cdot\sqrt{5}},$$

$$=\frac{\sqrt{10}+\sqrt{15}}{5}, \text{ or } \tfrac{1}{5}(\sqrt{10}+\sqrt{15}).$$

233. In the preceding example we have reduced the fraction to an equivalent fraction with a rational denominator. The process of rendering a quantity rational is called **rationalization.**

234. The factor by which an expression is multiplied to produce a rational expression is called a **rationalizing factor.**

E.g., $\sqrt{8}$ can be rationalized by multiplying it by $\sqrt{2}$, for $\sqrt{8}\cdot\sqrt{2}=\sqrt{16}=4$.

235. Since *the problem of division by surds reduces to that of the rationalization of the divisor*, exercises in rationalization will first be considered.

ILLUSTRATIVE PROBLEMS

1. By what expression may $a^{\frac{2}{3}}b^{\frac{1}{4}}$ be multiplied in order that the product shall be rational.

Evidently $\qquad a^{\frac{2}{3}} \cdot a^{\frac{1}{3}} = a,$

and $\qquad b^{\frac{1}{4}} \cdot b^{\frac{3}{4}} = b;$

hence $a^{\frac{1}{3}}b^{\frac{3}{4}}$ is a rationalizing factor of $a^{\frac{2}{3}}b^{\frac{1}{4}}$.

2. By what expression may $\sqrt[5]{a^4} \cdot \sqrt[4]{b^5}$ be multiplied in order that the product shall be rational?

1. $\sqrt[5]{a^4} \cdot \sqrt[4]{b^5} = a^{\frac{4}{5}}b^{\frac{5}{4}} = a^{\frac{4}{5}}b^{\frac{1}{4}}b.$

2. If, now, this is multiplied by $a^{\frac{1}{5}}b^{\frac{3}{4}}$ the product will evidently have no fractional exponents.

3. $\therefore\ a^{\frac{1}{5}}b^{\frac{3}{4}}$ is a rationalizing factor of $\sqrt[5]{a^4} \cdot \sqrt[4]{b^5}$.

In general, *a monomial with a proper fraction for an exponent can be rationalized by multiplying it by the monomial with an exponent equal to* 1 *less that fraction.*

I.e., $a^{1-\frac{1}{3}}$ is a rationalizing factor for $a^{\frac{1}{3}}$.

Of course there is no limit to the number of rationalizing factors; for if $a^{\frac{2}{3}}$ is a rationalizing factor for $a^{\frac{1}{3}}$, so are $a \cdot a^{\frac{2}{3}}$, $a^2 \cdot a^{\frac{2}{3}}$, $\cdots$. But evidently $a^{\frac{2}{3}}$ is the *simplest* one.

3. By what expression may $a + \sqrt{b}$ be multiplied in order that the product shall be rational?

1. $\because\ (x - y)(x + y) = x^2 - y^2,$ § 68

2. $\therefore\ (a - \sqrt{b})(a + \sqrt{b}) = a^2 - b$, a rational expression.

3. $\therefore\ a - \sqrt{b}$ is a rationalizing factor.

Similarly, $a + \sqrt{b}$ is a rationalizing factor for $a - \sqrt{b}$.

236. And, in general, *the conjugate of a binomial quadratic surd* (§ 68, 3) *is a rationalizing factor of that surd.*

EXERCISE XCV

Find the simplest rationalizing factor for each of the following expressions:

1. $\sqrt{2}$.
2. $\sqrt[3]{4}$.
3. $\sqrt{ab}$.
4. $\sqrt[3]{ab^2c}$.
5. $a^{\frac{7}{8}}b^{\frac{5}{6}}c^{\frac{3}{2}}$.
6. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$.
7. $\sqrt[7]{a^5b^4c^3}$.
8. $\sqrt[5]{x^2y^3z^4}$.
9. $2+\sqrt{3}$.
10. $3-\sqrt{2}$.
11. $\sqrt{5}-1$.
12. $a^3+\sqrt{b}$.
13. $a^2-\sqrt{b}$.
14. $a^{\frac{1}{16}}b^{\frac{7}{8}}c^{\frac{n-1}{n}}$.
15. $\sqrt{a}+1$.
16. $\sqrt{a}-\sqrt{3}$.
17. $\sqrt{7}+\sqrt{5}$.
18. $\sqrt{9}+\sqrt{4}$.
19. $\sqrt{a}-\sqrt{b}$.
20. $\sqrt{7}-\sqrt{5}$.
21. $\sqrt{5}+\sqrt{7}$.
22. $\sqrt{5}-\sqrt{2}$.
23. $\sqrt{2}-\sqrt{22}$.
24. $\sqrt{2}+\sqrt{7}$.
25. $\sqrt{a+b}+c$.
26. $\sqrt{a+b}+\sqrt{a-b}$.
27. $\sqrt{\frac{1}{x^2}+x^2+2}-\sqrt{x}$.
28. $\sqrt{a+b+c}+d+e$.
29. $\sqrt{a+b+c}-\sqrt{a-b-c}$.
30. $\sqrt{1+2+6}+\sqrt{7+8+10}$.
31. $\sqrt{a-b^2+c^3}-\sqrt{a+b^2-c^3}$.
32. $\sqrt{x^2+y^2+z^2}-\sqrt{x^2-y^2+z^2}$.
33. $\sqrt{a^2b^2c^2+d^2e^2f^2}+\sqrt{a^2b^2c^2-d^2e^2f^2}$.
34. $\sqrt{a^2+b^2+2ab}+\sqrt{a^2+b^2-2ab}$.

ILLUSTRATIVE PROBLEMS IN DIVISION

1. Divide $\sqrt[3]{12}$ by $\sqrt[3]{3}$.

$$\frac{\sqrt[3]{12}}{\sqrt[3]{3}}=\sqrt[3]{\frac{12}{3}}=\sqrt[3]{4}. \qquad \S\ 205$$

2 Divide $\sqrt{5}$ by $\sqrt[3]{2}$.

1. $$\frac{\sqrt{5}}{\sqrt[3]{2}}=\frac{\sqrt[6]{5^3}}{\sqrt[6]{2^2}}=\sqrt[6]{\frac{5^3}{2^2}} \qquad \S\ 205$$

2. $$=\sqrt[6]{\frac{2^4\cdot 5^3}{2^6}}=\tfrac{1}{2}\sqrt[6]{2000}.$$

3. Divide $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$. That is, rationalize the denominator of the fraction $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

1. The rationalizing factor for the denominator is evidently
$$\sqrt{2}+\sqrt{3}.$$

2. $$\frac{(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}=\frac{2+2\sqrt{6}+3}{2-3}=-(5+2\sqrt{6}).$$

EXERCISE XCVI

Perform the divisions indicated in Exs. 1–26, or rationalize the divisor.

1. $2:\sqrt{3}$.
2. $5:\sqrt[3]{2a}$.
3. $6:4\sqrt[4]{24}$.
4. $3:\sqrt{3xy}$.
5. $2\sqrt{x}:\sqrt{y}$.
6. $4ab:\sqrt[4]{ab}$..
7. $2x^2:\sqrt[3]{x^2y}$.
8. $a\sqrt{b}:b\sqrt{a}$.
9. $ab^2c:\sqrt[3]{ab^2c}$.
10. $2ax:\sqrt{3a^2x^2}$.
11. $3a^2b^3:2\sqrt[3]{a^2b}$.
12. $15\sqrt{24}:3\sqrt[3]{24}$.
13. $24:(2\sqrt{7}-6)$.
14. $58:(8+\sqrt{35})$.
15. $3\sqrt[3]{xy^2}:4\sqrt[4]{x^3y^2}$.
16. $12\sqrt[3]{192}:4\sqrt[5]{729}$.
17. $16\sqrt[3]{a^2bc^2}:8\sqrt[4]{a^3b^2c}$.
18. $90:(5\sqrt{3}-\sqrt{30})$.

19. $10\sqrt{12} : 2\sqrt{18} : 4\sqrt{8}$.

20. $(18 - 16\sqrt{5}) : (4 - \sqrt{5})$.

21. $(\sqrt{12} - \sqrt{18} + \sqrt{6}) : \sqrt{2}$.

22. $\sqrt[4]{(a^3 - 2b)^3} : \sqrt{(a^3 - 2b)}$.

23. $(3\sqrt{5} - 8\sqrt{2}) : (3\sqrt{3} - 4\sqrt{5})$.

24. $(3\sqrt{3} - 2\sqrt{2}) : (5\sqrt{8} - 3\sqrt{2})$.

25. $(7\sqrt{12} - 4\sqrt{27}) : (8\sqrt{3} + 2\sqrt{2})$.

26. $(15\sqrt{8} + 10\sqrt{7} - 8\sqrt{2} + 5) : -4\sqrt{5}$.

Rationalize the denominators of the fractions in Exs. 27–40.

27. $\dfrac{30}{2 - 3^{\frac{1}{2}}}$.

28. $\dfrac{3 - 5^{\frac{1}{2}}}{3 + 5^{\frac{1}{2}}}$.

29. $\dfrac{3 - \sqrt{7}}{3 + \sqrt{7}}$.

30. $\dfrac{a}{a + \sqrt{b}}$.

31. $\dfrac{3 + \sqrt[3]{3}}{3 + \sqrt{3}}$.

32. $\dfrac{2 + \sqrt{5}}{3 - \sqrt{7}}$.

33. $\dfrac{2 - \sqrt{8}}{8 - \sqrt{2}}$.

34. $\dfrac{11^{\frac{1}{2}} + 5^{\frac{1}{2}}}{11^{\frac{1}{2}} - 5^{\frac{1}{2}}}$.

35. $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$.

36. $\dfrac{7 + 3\sqrt{7}}{12 - 6\sqrt{11}}$.

37. $\dfrac{a + b + \sqrt{a + b}}{a - b + \sqrt{a - b}}$.

38. $\dfrac{2}{(a^2 + b)^{\frac{1}{2}} + (a^2 - b)^{\frac{1}{2}}}$.

39. $\dfrac{2m}{(a + m)^{\frac{1}{2}} + (a - m)^{\frac{1}{2}}}$.

40. $\dfrac{1}{x(1 - a^2)^{\frac{1}{2}} - y(1 + a^2)^{\frac{1}{2}}}$.

237. The approximate values of certain surd expressions may be obtained by applying the principles already discovered.

For example, required the approximate value of $\frac{\sqrt{5}}{\sqrt{7}}$.

Here we may say

$$\frac{\sqrt{5}}{\sqrt{7}}=\sqrt{\frac{5}{7}}=\sqrt{0.71428\cdots},$$

and then extract the root.

Or we may say

$$\frac{\sqrt{5}}{\sqrt{7}}=\frac{2.236068\cdots}{2.645751\cdots},$$

and then divide.

But it is much easier to say

$$\frac{\sqrt{5}}{\sqrt{7}}=\tfrac{1}{7}\sqrt{35},$$

and then extract the root and divide.

$$\begin{aligned}\therefore\ \frac{\sqrt{5}}{\sqrt{7}}=\tfrac{1}{7}\sqrt{35}&=\tfrac{1}{7}\text{ of }5.9160798.\\&=0.8451543.\end{aligned}$$

EXERCISE XCVII

Find, by the method recommended in § 237, the approximate values (to 4 decimal places) of the following:

1. $\frac{5}{\sqrt{3}}$.
2. $\frac{3}{\sqrt{2}}$.
3. $\frac{\sqrt{2}}{\sqrt{7}}$.
4. $\frac{7}{\sqrt{2}}$.
5. $10\frac{\sqrt{5}}{\sqrt{3}}$.
6. $\frac{3}{\sqrt{5}}$.
7. $\frac{2}{2-\sqrt{3}}$.
8. $\frac{2-\sqrt{3}}{2+\sqrt{3}}$.
9. $\frac{\sqrt{2}-1}{\sqrt{2}+1}$.

238. Roots of surds. — The roots of perfect powers of surd expressions can often be found by inspection or extracted in the ordinary way.

1. To find the square root of $a + 4\sqrt{ab} + 4b$.

1. $\because \quad \sqrt{f^2 + 2fn + n^2} = \pm (f + n),$

2. $\therefore \quad \sqrt{a + 4\sqrt{ab} + 4b} = \pm (\sqrt{a} + 2\sqrt{b}).$

Check. $\sqrt{9} = \pm 3.$

2. To find the fifth root of the perfect fifth power

$$a^{\frac{5}{2}} - 5a^2b^{\frac{1}{3}} + 10a^{\frac{3}{2}}b^{\frac{2}{3}} - 10ab + 5a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}.$$

By comparing this with the expansion of $(a + b)^5$, in § 182, it is readily seen to be the 5th power of $a^{\frac{1}{2}} - b^{\frac{1}{3}}$.

3. To find the square root of $7 + 4\sqrt{3}$.

1. If this can be brought into the form $f^2 + 2fn + n^2$, the root will be in the form $\pm (f + n)$.

2. We first make the coefficient of the second term 2, because of the $2fn$, and have $7 + 2\sqrt{12}$.

3. And $\because$ 12 is the product of 3 and 4, and 7 is the sum of 3 and 4, we may write this

$$4 + 2\sqrt{4 \cdot 3} + 3,$$

which is evidently the square of

$$\sqrt{4} + \sqrt{3},$$

or of

$$2 + \sqrt{3}.$$

4. To find the square root of $8 - 2\sqrt{15}$.

1. As in Ex. 3 we attempt to bring this into the form $f^2 + 2fn + n^2$.

2. $\because$ 15 is the product of 5 and 3, and 8 is their sum, we may write this

$$5 - 2\sqrt{5 \cdot 3} + 3,$$

which is evidently the square of

$$\sqrt{5} - \sqrt{3}.$$

239. Although these square roots have the double sign ($\pm$), it is customary in elementary work to consider only the positive value (§ 225).

EXERCISE XCVIII

Extract the square roots in Exs. 1–6.

1. $a - 2a^{\frac{3}{2}} + a^2.$

2. $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{2}{3}} + b^{\frac{4}{3}}.$

3. $a - 2\sqrt{2ab} + 2b.$

4. $2a - \sqrt{200a} + 25.$

5. $3a - 8\sqrt{3a} + 16.$

6. $x^4 + 2x^2\sqrt[3]{y} + \sqrt[3]{y^2}.$

Extract the cube roots in Exs. 7–10.

7. $8 - 12\sqrt{a} + 6a - a\sqrt{a}.$

8. $a - 3\sqrt[3]{a^2b^2} + 3b\sqrt[3]{ab} - b^2.$

9. $x^3 - 3x^2\sqrt[4]{y} + 3x\sqrt{y} - \sqrt[4]{y^3}.$

10. $x^4\sqrt{x} - 3x^3\sqrt[3]{y} + 3x\sqrt[6]{x^3y^4} - y.$

Extract the fifth roots in Exs. 11, 12.

11. $1 - 5y^{\frac{2}{5}} + 10y^{\frac{4}{5}} - 10y^{\frac{6}{5}} + 5y^{\frac{8}{5}} - y^2.$

12. $32 - 80\sqrt[5]{y^2} + 80\sqrt[5]{y^4} - 40y\sqrt[5]{y} + 10y\sqrt[5]{y^3} - y^2.$

Extract the square roots in Exs. 13–30.

13. $\frac{3}{2} + \sqrt{2}.$

14. $9 - 4\sqrt{2}.$

15. $8 - 2\sqrt{7}.$

16. $\frac{5}{6} + \frac{1}{3}\sqrt{6}.$

17. $4 + 2\sqrt{3}.$

18. $7 - 4\sqrt{3}.$

19. $8 + \sqrt{60}.$

20. $10 - \sqrt{96}.$

21. $18 + 8\sqrt{5}.$

22. $7 + 2\sqrt{10}.$

23. $11 + 2\sqrt{30}.$

24. $10\sqrt{7} + 32.$

25. $112 + 40\sqrt{3}.$

26. $75 - 12\sqrt{21}.$

27. $2x + 2\sqrt{x^2 - 1}.$

28. $2x + 2\sqrt{x^2 - y^2}.$

29. $ab - 2a\sqrt{ab - a^2}.$

30. $x^2 + x + y + 2x\sqrt{x + y}.$

REVIEW EXERCISE XCIX

1. Simplify $\dfrac{a^m + b^n}{a^{-m} + b^{-n}} \cdot \dfrac{a^n - b^m}{a^{-n} - b^{-m}}$.

2. Simplify $(3^{\frac{3}{4}} + 3^{\frac{1}{2}} + 3^{\frac{1}{4}} + 1)(3^{\frac{1}{4}} - 1)$.

3. Simplify $3(a^{\frac{1}{2}}+x^{\frac{1}{2}})^2-4(a^{\frac{1}{2}}+x^{\frac{1}{2}})(a^{\frac{1}{2}}-x^{\frac{1}{2}})+(a^{\frac{1}{2}}-2\,x^{\frac{1}{2}})^2$.

4. Divide $x^{\frac{1}{2}}-4\,x^{\frac{3}{8}}a^{\frac{1}{8}}+6\,x^{\frac{1}{4}}a^{\frac{1}{4}}-4\,x^{\frac{1}{8}}a^{\frac{3}{8}}+a^{\frac{1}{2}}$ by $x^{\frac{1}{4}}-2\,x^{\frac{1}{8}}a^{\frac{1}{8}}$ $+\,a^{\frac{1}{4}}$.

Simplify the expressions in Exs. 5–11.

5. $2\,\sqrt[3]{8}\cdot\sqrt{\dfrac{1}{25}}\cdot 5\sqrt[3]{\dfrac{-3}{24}}\cdot\sqrt[5]{243}$.

6. $5\sqrt[3]{(a + 2\,b)^2}\,.\,\sqrt[3]{(a + 2\,b)^2}$.

7. $5\sqrt[3]{a(x + y)^2}\cdot 4\sqrt{a^3(x + y)}$.

8. $\sqrt{a + b}\cdot\sqrt{a - b}\cdot\sqrt{a^2 - b^2}$.

9. $ab\sqrt{ab}\cdot a^2b^2\sqrt[3]{a^2b^2}\cdot a^3b^3\sqrt[4]{a^3b^3}$.

10. $(a + b)\sqrt{a + b}\cdot(a - b)\sqrt{a - b}$.

11. $\sqrt{2}\cdot\sqrt{24\frac{1}{2}}\cdot\frac{1}{5}\sqrt{625}\cdot 7^{-1}\cdot 25^{-2}\cdot\sqrt[3]{-125}$.

Extract the square roots in Exs. 12–17.

12. $4\,a^{-\frac{1}{2}} + 4 + a^{\frac{1}{2}}$.

13. $a^{\frac{4}{3}} - 2\,a^{\frac{3}{3}} + 5\,a^{\frac{2}{3}} - 4\,a^{\frac{1}{3}} + 4$.

14. $a^2b^{-1} + \frac{1}{4}\,a^{-1}b^2 + (2\,a^{\frac{3}{2}} - b^{\frac{3}{2}})(ab)^{-\frac{1}{2}}$.

15. $x + y + z + 2\,x^{\frac{1}{2}}y^{\frac{1}{2}} - 2\,x^{\frac{1}{2}}z^{\frac{1}{2}} - 2\,y^{\frac{1}{2}}z^{\frac{1}{2}}$.

16. $a^2 + 4\,a^{\frac{3}{2}}y^{\frac{1}{3}} + 10\,ay^{\frac{2}{3}} + 12\,a^{\frac{1}{2}}y + 9\,y^{\frac{4}{3}}$.

17. $25\,x^{-4} - 30\,x^{-3}y + 49\,x^{-2}y^2 - 24\,x^{-1}y^3 + 16\,y^4$.

Extract the cube roots in Exs. 18–24.

18. $\frac{1}{8}x^{\frac{3}{2}}-\frac{3}{4}x+1\frac{1}{2}x^{\frac{1}{2}}-1.$

19. $\frac{1}{27}x^{-6}+\frac{2}{3}x^{-4}+4x^{-2}+8.$

20. $27x^{-3}-54x^{-2}y^{-2}+36x^{-1}y^{-4}-8y^{-6}.$

21. $8x^{2}+48x^{\frac{5}{3}}+60x^{\frac{4}{3}}-80x-90x^{\frac{2}{3}}+108x^{\frac{1}{3}}-27.$

22. $x^{-6}-9x^{-5}+33x^{-4}-63x^{-3}+66x^{-2}-36x^{-1}+8.$

23. $\frac{1}{64}x^{-\frac{3}{5}}+\frac{3}{32}x^{-\frac{1}{2}}+\frac{3}{8}x^{-\frac{2}{5}}+\frac{7}{8}x^{-\frac{3}{10}}+\frac{3}{2}x^{-\frac{1}{5}}+\frac{3}{2}x^{-\frac{1}{10}}+1.$

24. $8a^{\frac{3}{4}}+48a^{\frac{5}{8}}b+60a^{\frac{1}{2}}b^{2}-80a^{\frac{3}{8}}b^{3}-90a^{\frac{1}{4}}b^{4}+108a^{\frac{1}{8}}b^{5}-27b^{6}.$

25. Simplify $\sqrt{21}\cdot\sqrt{\frac{3}{7}}.$

26. Factor $6x^{2}-19x+15.$

27. Add $\dfrac{x+xy}{x-xy}$ and $\dfrac{y+y^{2}}{y-y^{2}}.$

28. Factor $x^{4}-(a-b)x^{2}-ab.$

29. Simplify $\dfrac{\sqrt[3]{x^{-4}}\cdot\sqrt{x^{-5}\cdot x^{\frac{1}{3}}}}{x^{-\frac{5}{3}}\cdot\sqrt[4]{x^{10}x^{-\frac{1}{3}}}}.$

30. Simplify $5\sqrt{24}-2\sqrt{54}-\sqrt{6}.$

31. Simplify $\dfrac{(x^{\frac{3}{2}}y^{-\frac{1}{2}}z^{2})^{\frac{3}{4}}\cdot(x^{-\frac{1}{2}}yz^{\frac{1}{4}})}{(xy^{2})^{-\frac{3}{4}}\cdot(y^{2}z)^{\frac{1}{2}}\cdot(z^{\frac{1}{2}}x^{-\frac{1}{2}})^{\frac{1}{4}}}.$

32. Simplify $a-\{b-[a-(b-x)]\}.$

33. Simplify $\dfrac{\sqrt[5]{x^{3}y^{2}z}\cdot\sqrt[5]{y^{3}z^{2}x}\cdot\sqrt[5]{z^{3}x^{2}y}}{\sqrt[3]{\frac{x^{5}}{yz}}\cdot\sqrt[3]{\frac{y^{5}}{zx}}\cdot\sqrt[3]{\frac{z^{5}}{xy}}}.$

34. Simplify $\sqrt[3]{x^{2}y^{2}}\cdot\sqrt{x^{3}y^{3}}\cdot\sqrt{x}\cdot\sqrt{y}.$

35. Simplify $\dfrac{2\sqrt{3}}{\sqrt{5}+2\sqrt{3}}+\dfrac{\sqrt{5}}{2\sqrt{3}-\sqrt{5}}$.

36. Multiply $x+\sqrt{3}-1$ by $x-\sqrt{3}-1$.

37. Factor $(x-1)^3+(x-2)^3+(3-2x)^3$.

38. Extract the square root of $37+4\sqrt{78}$.

39. Rationalize the denominator of $\dfrac{1}{\sqrt{3}-1}$.

40. Multiply $x^{p+q}+x^p+1$ by $x^{p-q}-x^p+1$.

41. Solve the equations $\dfrac{3}{x}-\dfrac{1}{y}=4$, $\dfrac{1}{x}-\dfrac{2}{y}=-2$.

42. Simplify $(\sqrt{x}+\sqrt{y})(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y})$.

43. Find the highest common factor of

$x^4+5x^3+11x^2+13x+6$ and x^3-x^2-3x-9.

44. Extract the square root of $(2x-3a)^2-4(2x-3a)+4$.

45. Show that $\sqrt{a+b}+\sqrt{a-b}=\sqrt{2}\cdot\sqrt{a+\sqrt{a^2-b^2}}$.

46. Find the cube root of $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$.

47. Solve the equation $(x+1)^2+(x+4)^2=(x+2)^2+(x+5)^2$.

48. What is the remainder when $2x^5-2x^4+3x^3-7x^2+5x-8$ is divided by $x+2$?

49. If the numerator of a certain fraction be doubled, and its denominator increased by 7, it becomes $\frac{1}{2}$; if the denominator be doubled and the numerator increased by 7, it becomes unity. Required the fraction.

CHAPTER XIII

COMPLEX NUMBERS

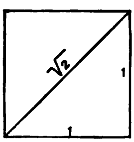
I. DEFINITIONS

240. Certain steps in the growth of the number system have already been set forth in § 24, but are here repeated because they lead up to the *imaginary number.*

1. **The positive integer** suffices for the solution of the equation $x - 3 = 0$, since $x = 3$ satisfies the equation. We can represent such a number by a line three units long, as in the annexed figure, the unit being of any convenient length.

2. **The positive fraction.** If, however, we attempt to solve the equation $3x - 2 = 0$, either we must say that the solution is impossible or we must extend the idea of number to include the positive fraction. Then $x = \frac{2}{3}$ satisfies the equation. We can represent such a number by dividing a line one unit long into three parts and taking two of them.

3. **The surd.** If we attempt to solve the equation $x^2 - 2 = 0$, either we must say that the solution is impossible or we must extend the idea of number to include the surd. Then $\sqrt{2}$ satisfies the equation. We can represent $\sqrt{2}$ by the diagonal of a square whose side is one unit long. This is evident because the square on the hypotenuse equals the sum of the squares on the two sides of the right-angled triangle.



4. **The negative number.** If we attempt to solve the equation $x+2=0$, either we must say that the solution is impossible or we must extend the idea of number to include the negative number. Then $x=-2$ satisfies the equation. We can represent such a number by supposing the negative sign to denote direction, a direction opposite to that which we assume for positive numbers.

241. The numbers thus far described in this chapter are called **real numbers**.

242. **The imaginary number.** If we attempt to solve the equation $x^2+1=0$, either we must say that the solution is impossible or we must extend the idea of number still further.

The equation $\qquad x^2+1=0$

leads to $\qquad x^2=-1,$

which leads to $\qquad x=\pm\sqrt{-1},$

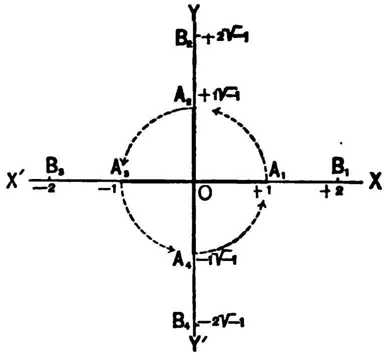
which cannot be a positive or a negative integer, fraction, or surd (§ 188).

243. We call an even root of a negative number an *imaginary number.*

The term "imaginary" is unfortunate, since these numbers are no more imaginary than are fractions or negative numbers. We cannot imagine looking out of a window -2 times or $\frac{1}{2}$ of a time any more than $\sqrt{-1}$ times. The "imaginary" is merely another step in the number system. The name is, however, so generally used that it should continue to designate this new form of number.

To the ancients, negative numbers were as "imaginary" as $\sqrt{-1}$ is to us. It was only when some one drew a picture of $\sqrt{2}$ (see § 240, 3), of -1, and later of $\sqrt{-1}$, that these were understood.

244. As with fractions, surds, and negative numbers, so it is necessary to represent the imaginary graphically by a line, or in some other concrete way, in order to make its nature clear to the beginner.



In this figure the multiplication of $+1$ by -1 swings the line OA_1 through 180° to the position OA_3.

As a matter of custom this line is supposed to swing as indicated by the arrows, opposite to the movement of clock-hands, *counter-clockwise*.

245. That is, since $(\sqrt{-1})^2$ means $\sqrt{-1}\cdot\sqrt{-1}$ or -1, the multiplication of $+1$ by $\sqrt{-1}\cdot\sqrt{-1}$ swings $+1$ through 180°; therefore the multiplication of $+1$ by $\sqrt{-1}$ should be regarded as swinging it through half of this angle, or 90°, to the position OA_2.

Or we may say that since multiplication by $\sqrt{-1}$ twice, carries OA_1 through 180°, therefore multiplication by $\sqrt{-1}$ once should carry it through 90°.

246. Hence, *we represent* $+1\sqrt{-1}$ (*or* $+\sqrt{-1}$), $+2\sqrt{-1}$, $+3\sqrt{-1}$, $\cdots$, *by integers on the perpendicular* OY, upward from O, and $-1\sqrt{-1}$ (*or* $-\sqrt{-1}$), $-2\sqrt{-1}$, $-3\sqrt{-1}$, $\cdots$, *by integers on the negative side of this line*, i.e., on OY', downward from O.

It is thus seen that as positive numbers are represented to the right of the zero point, and negative numbers to the left, so positive imaginaries are represented above the zero point and negative imaginaries below.

247. Hence, it appears that *the symbols* $+\sqrt{-1}$ *and* $-\sqrt{-1}$ *are, like* $+$ *and* $-$, *symbols of quality* and may be looked upon as indicating direction.

E.g.,	$+3$	indicates 3 units to the right,
	-3	" " " left,
	$+3\sqrt{-1}$	" " up,
	$-3\sqrt{-1}$	" " down.

248. Since $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$, we say that $\sqrt{-3}$ shall equal $\sqrt{3\cdot -1}=\sqrt{3}\cdot\sqrt{-1}$. Hence numbers of the form $\sqrt{-m}$ may always be written in the form $a\sqrt{-1}$, where a is real (possibly a surd or a fraction).

ILLUSTRATIVE PROBLEMS

1. Represent graphically $\sqrt{-9}$.

$$\sqrt{-9}=3\sqrt{-1}.$$

Hence we measure 3 units upwards on the perpendicular OY.

2. Solve the equation $x^2+3=0$, expressing the result in the form $a\sqrt{-1}$.

$$x^2+3=0.$$
$$\therefore\ x^2=-3.$$
$$\therefore\ x=\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}.$$

3. Solve the equation $3x^2+7=0$, expressing the result in the form $a\sqrt{-1}$.

$$3x^2+7=0.$$
$$\therefore\ x^2=-\tfrac{7}{3}.$$
$$\therefore\ x=\sqrt{-\tfrac{7}{3}}$$
$$=\tfrac{1}{3}\sqrt{-7\cdot 3}$$
$$=\tfrac{1}{3}\sqrt{21}\cdot\sqrt{-1}.$$

EXERCISE C

Solve the following equations, expressing the results in the form $a\sqrt{-1}$:

1. $x^2=-9$.
2. $3x^2+2=0$.
3. $5x^2=-5$.
4. $x^2\sqrt{2}=-3$.
5. $x^2+5=0$.
6. $5x^2=-125$.
7. $x^2+4=0$.
8. $x^2+20=-5$.
9. $x^2+20=4$.
10. $x^2-10=-46$.
11. $3x^2-1=x^2-5$.
12. $(x+1)^2=2x-10$.

Represent graphically the following imaginary numbers:

13. $\sqrt{-4}$.
14. $\sqrt{-5}$.
15. $-5\sqrt{-1}$.
16. $\sqrt{-32}$.
17. $3\sqrt{-1}$.
18. $\sqrt{2}\cdot\sqrt{-2}$.
19. $-\sqrt{-16}$.
20. $2\sqrt{-9}$.
21. $-\frac{1}{2}\sqrt{-12}$.

249. The complex number. If we attempt to solve the equation $x^2-4x+5=0$ by factoring, we may write it in the form

$$x^2-4x+4-(-1)=0,$$

or
$$(x-2)^2-(-1)=0,$$

or
$$(x-2+\sqrt{-1})(x-2-\sqrt{-1})=0,$$

whence
$$x=2-\sqrt{-1},$$

or
$$x=2+\sqrt{-1}.$$

Hence, it appears that each root is the algebraic sum of a real number and an imaginary.

Such a number is said to be **complex**.

In general, a number of the form $a+b\sqrt{-1}$ is a complex number.

II. OPERATIONS WITH IMAGINARY AND COMPLEX NUMBERS

250. Imaginary and complex numbers obey the laws of other irrational numbers. The only difficulty which the student meets is in the multiplication of two imaginaries. By always writing these, if possible, in the form $a\sqrt{-1}$, and remembering that $\sqrt{-1}\cdot\sqrt{-1} = -1$, this difficulty will be avoided.

ILLUSTRATIVE PROBLEMS

1. Add $\sqrt{-4}$, $\sqrt{-16}$, and $\sqrt{-5}$.

$$\begin{array}{r}
\sqrt{-4} = \quad 2\sqrt{-1}\\
\sqrt{-16} = \quad 4\sqrt{-1}\\
\underline{\sqrt{-5} = \sqrt{5}\sqrt{-1}}\\
(6+\sqrt{5})\sqrt{-1}.
\end{array}$$

2. From $\sqrt{-7}$ take $\sqrt{-5}$.

$$\begin{array}{r}
\sqrt{-7} = \sqrt{7}\cdot\sqrt{-1}\\
\underline{\sqrt{-5} = \sqrt{5}\cdot\sqrt{-1}}\\
(\sqrt{7}-\sqrt{5})\sqrt{-1}.
\end{array}$$

3. Add $3+\sqrt{-3}$, $5-2\sqrt{-4}$, $-4+3\sqrt{-9}$.

$$\begin{array}{r}
3+\sqrt{3}\sqrt{-1}\\
5- \quad 4\sqrt{-1}\\
\underline{-4+ \quad 9\sqrt{-1}}\\
4+(5+\sqrt{3})\sqrt{-1}.
\end{array}$$

4. Multiply $2+\sqrt{-3}$ by $3+\sqrt{-2}$.

$$\begin{array}{l}
2+\ \sqrt{3}\sqrt{-1}\\
\underline{3+\ \sqrt{2}\sqrt{-1}}\\
6+3\sqrt{3}\sqrt{-1}\\
\underline{\quad +2\sqrt{2}\sqrt{-1}+\sqrt{6}\cdot -1}\\
6+(3\sqrt{3}+2\sqrt{2})\sqrt{-1}-\sqrt{6}.
\end{array}$$

5. Divide $2+\sqrt{-3}$ by $2-\sqrt{-3}$.

Multiplying each term of the fraction by $2+\sqrt{-3}$, we have

$$\frac{2+\sqrt{-3}}{2-\sqrt{-3}}=\frac{(2+\sqrt{3}\sqrt{-1})^2}{(2+\sqrt{3}\sqrt{-1})(2-\sqrt{3}\sqrt{-1})}$$

$$=\frac{4+4\sqrt{3}\sqrt{-1}+3\cdot-1}{4-3\cdot-1}$$

$$=\frac{1+4\sqrt{-3}}{7}.$$

6. Raise $\sqrt{-1}$ to the 2d, 3d, 4th, and 5th powers.

$$(\sqrt{-1})^2=-1;$$

$$(\sqrt{-1})^3=-1\sqrt{-1}=-\sqrt{-1};$$

$$(\sqrt{-1})^4=[(\sqrt{-1})^2]^2=(-1)^2=1;$$

$$(\sqrt{-1})^5=(\sqrt{-1})^4\cdot\sqrt{-1}=\sqrt{-1}.$$

Hence the 5th power is the same as the first power.

EXERCISE CI

Perform the operations indicated.

1. $(3+\sqrt{-2})(4+\sqrt{-2})$.
2. $\sqrt{-25}-\sqrt{16}-\sqrt{-1}$.
3. $(5-\sqrt{-3})(4+\sqrt{-3})$.
4. $\sqrt{-4}+\sqrt{-9}+\sqrt{-16}$.
5. $2+\sqrt{-4}-(2-\sqrt{-4})$.
6. $(3-4\sqrt{-1})(5+\sqrt{-2})$.
7. $\sqrt{-5}+\sqrt{-7}+\sqrt{-28}$.
8. $\sqrt{-36}-\sqrt{-9}+\sqrt{-25}$.
9. $\sqrt{-7}+\sqrt{-28}-\sqrt{-175}$.

10. $5 \div (2 - \sqrt{-1})$.

11. $7 \div (2 - \sqrt{-3})$.

12. $74 \div (5 + \sqrt{-7})$.

13. $58 \div (3 - 7\sqrt{-1})$.

14. $53\sqrt{-1} \div (7 + 2\sqrt{-1})$.

15. $(2 + \sqrt{-81})(9 + 2\sqrt{-1})$.

16. $(8 + 7\sqrt{-1})(8 - \sqrt{-49})$.

17. $(3 + 4\sqrt{-4})(4 + 3\sqrt{-3})$.

18. $(2 - 7\sqrt{-1})(7 - 2\sqrt{-1})$.

19. $(3 + 2\sqrt{-1})^2(3 - 2\sqrt{-1})$.

20. $\sqrt{-5} + \sqrt{-125} - \sqrt{-45}$.

21. $(2 + \sqrt{-25}) \div (\sqrt{-4} - 1)$.

22. $(8 - \sqrt{-4}) \div 2(2 + \sqrt{-1})$.

23. $(11 + \sqrt{-49}) \div (4 + \sqrt{-1})$.

24. $(14 + \sqrt{-64}) \div (5 + \sqrt{-1})$.

25. $\sqrt{-49} + \sqrt{-144} - \sqrt{-25}$.

26. $\sqrt{-11} + \sqrt{-44} - \sqrt{-275}$.

27. $\sqrt{-13} - \sqrt{-65} - \sqrt{-117}$.

28. $(2\sqrt{-11} - 7) \div (2 + \sqrt{-11})$.

29. $\sqrt{-1} + \sqrt{-2} + \sqrt{-3} + \sqrt{-4}$.

30. $(2 + \sqrt{3})\sqrt{-1} + (3 + \sqrt{3})\sqrt{-1}$.

31. $[30 - 3\sqrt{2} + (5\sqrt{3} + 6\sqrt{6})\sqrt{-1}] \div (5 + \sqrt{-6})$.

32. Show that the 6th power of $\sqrt{-1}$ is the same as the 2d power, and the 7th as the 3d.

33. Cube $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and show that the result is the same as the cube of $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ and the cube of 1.

REVIEW EXERCISE CII

1. Divide $x^{-1} - y^{-1}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$.

2. Reduce to lowest terms the fraction

$$\frac{(x^4 - y^4)(x^3 - y^3)}{(x^6 - y^6)(x - y)}.$$

3. Solve the set of equations

$$3x + 4y - 1 = 0,$$
$$18x + 24y - 6 = 0.$$

4. Simplify $\dfrac{x^4 - x^2}{x^4 + x^3} \div \dfrac{x^2}{x^2+1} \div \dfrac{x-1}{x^2}$.

5. Add $\dfrac{a^2b - a^2c}{a}$, $\dfrac{b^2c - b^2a}{b}$, and $\dfrac{c^2a - c^2b}{c}$.

6. Factor $x^4 + y^4 + 1 - 2x^2y^2 - 2x^2 - 2y^2$.

7. Extract the square root of $19 - 4\sqrt{21}$.

8. Find the lowest common multiple of

$$x^3 + x - 2 \text{ and } x^4 + x^3 - 2x - 4.$$

9. Multiply $x^{\frac{3}{4}} - x^{\frac{1}{4}} + x^{-\frac{1}{4}} - x^{-\frac{3}{4}}$ by $x^{\frac{1}{2}} + 1 + x^{-\frac{1}{2}}$.

10. Find the positive integral roots of $3x + 2y = 24$.

11. Simplify

$$[x - (2x - 3y)] + [x - (2x + 3y)] - [x - (3y - 2x)].$$

12. Simplify $\left[\left(\dfrac{\sqrt{x}}{\sqrt[4]{x}}\right)^2 (\sqrt[4]{x} \cdot \sqrt{x})^4\right] \div \left[\left(\dfrac{\sqrt[4]{x}}{\sqrt{x}}\right)^2 \left(\dfrac{\sqrt{x}}{x}\right)^3\right]^2$.

13. Multiply together $1 + \dfrac{y}{x-y}$, $1 - \dfrac{2y}{x+y}$, and $1 + \dfrac{y}{x}$.

14. The length of a rectangle is three times its width. If each side be increased by 1 ft., its area is increased by 65 sq. ft. Required its dimensions.

CHAPTER XIV

RADICAL EQUATIONS

251. An equation in which the unknown quantity appears in surd form is called a **radical equation**.

E.g., $\sqrt{x+2}=5$ is a radical equation.

252. Radical equations can often be solved like simple equations, by *isolating the radical* and then raising to the proper power.

ILLUSTRATIVE PROBLEMS

1. Solve the equation $\sqrt{x-2}-\sqrt{x-5}=1$.

1. We first isolate the radical $\sqrt{x-2}$, by adding $\sqrt{x-5}$ to each member.

2. $$\therefore\ \sqrt{x-2}=1+\sqrt{x-5}.$$

3. Then, by squaring both members,

$$x-2=1+x-5+2\sqrt{x-5}.$$

4. Then, isolating the radical $\sqrt{x-5}$, by subtracting $1+x-5$ and dividing by 2,

$$1=\sqrt{x-5}.$$

5. $$\therefore\ 1=x-5, \text{ whence } x=6.$$

Check. $\sqrt{6-2}-\sqrt{6-5}=1$.

If the equation contains several irrational expressions, there is no general rule for solution. The student must use his judgment as to which radical it is best to isolate first.

2. Solve the equation

$$\sqrt{x+1}-4\sqrt{x-4}+5\sqrt{x-7}=0.$$

1. Isolating the radical $4\sqrt{x-4}$ by adding it to both members we have:

$$\sqrt{x+1}+5\sqrt{x-7}=4\sqrt{x-4}.$$

2. Squaring,

$$x+1+25x-175+10\sqrt{x^2-6x-7}=16x-64.$$

3. $$\therefore\ x-11=-\sqrt{x^2-6x-7}.$$

4. $$\therefore\ x^2-22x+121=x^2-6x-7.$$

5. $$\therefore\ x=8.$$

Check. $\sqrt{9}-4\sqrt{4}+5\sqrt{1}=3-8+5=0.$

EXERCISE CIII

Solve the following equations:

1. $\sqrt{x+5}=8.$

2. $\sqrt{x+a}=\frac{1}{2}\sqrt{x}.$

3. $\dfrac{\sqrt{x+5}}{\sqrt{x-2}}=1\frac{1}{3}.$

4. $\dfrac{\sqrt{x}-\sqrt{3}}{\sqrt{x}+\sqrt{3}}=\dfrac{x}{x-3}.$

5. $\dfrac{\sqrt{x}}{\sqrt{x+11}}=\dfrac{5}{6}.$

6. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\dfrac{x}{x-a}.$

7. $\dfrac{\sqrt{x+4}}{\sqrt{x-1}}=1.5.$

8. $\dfrac{\sqrt{x-4}}{\sqrt{x+3}}=\dfrac{\sqrt{x+23}}{\sqrt{x+51}}.$

9. $\dfrac{\sqrt{x+11}}{\sqrt{x+20}}=0.8.$

10. $\dfrac{\sqrt{x+1}}{4}=\dfrac{5}{\sqrt{x+10}}.$

11. $\sqrt{x}+\sqrt{x+7}=7.$

12. $\sqrt{x+2}+\sqrt{x-3}=5.$

13. $\sqrt{2x}+\sqrt{x-4}=6.$

14. $\sqrt{x+2}-\sqrt{x-3}=1.$

15. $\sqrt{x+1}+\sqrt{x-2}=3.$

16. $\sqrt{x-1}+\sqrt{x-6}=5.$

17. $\sqrt{x+5}-\sqrt{x-2}=1.$

18. $\sqrt{x}+3=\sqrt{5}+\sqrt{x+4}.$

19. $\sqrt{x}-2=\sqrt{7}-\sqrt{x-3}.$

20. $\sqrt{x+13}+\sqrt{x+24}=11.$

21. $\sqrt{x-7}+\sqrt{x+9}-8=0.$

22. $\sqrt{3x+4}-\sqrt{x+2}-2=0.$

23. $\sqrt{x+2}+\sqrt{x+3}=\sqrt{2x+5}.$

24. $\sqrt{x-1}+\sqrt{x-4}=\sqrt{2x-1}.$

REVIEW EXERCISE CIV

1. Add $\dfrac{x-a}{x+a}$, $\dfrac{x+a}{x-a}$, and $\dfrac{x^2-5a^2}{x^2-a^2}$.

2. Simplify $\dfrac{(x+y)^2-(x-y)^2}{(x+y)^2+(x-y)^2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$.

3. Reduce to lowest terms the fraction

$$\frac{x^2+p^2x+p^3-p^2}{x^2+(p^2-2p)x-p^3+p^2}.$$

4. Extract the square root of $14+8\sqrt{3}$.

5. Simplify $(14+6\sqrt{5})^{\frac{3}{2}}+(14-6\sqrt{5})^{\frac{3}{2}}$.

6. Simplify $\dfrac{4+\sqrt{2}}{2+\sqrt{2}}+\dfrac{4-\sqrt{2}}{3+\sqrt{2}}+\dfrac{9\sqrt{2}-13}{3-\sqrt{2}}$.

7. Find, to 3 decimal places, the value of $\dfrac{3}{\sqrt{7}}$.

8. Solve the equation $\sqrt{x+5}+1=\sqrt{x-2}+2.$

9. Solve the equation $\dfrac{2}{3x+2}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}$.

CHAPTER XV

QUADRATIC EQUATIONS INVOLVING ONE UNKNOWN QUANTITY

I. METHODS OF SOLVING

253. A **quadratic equation** (or equation of the second degree) involving one unknown quantity is an equation which can be reduced to the form $ax^2 + bx + c = 0$, a, b, c being known quantities and a not being zero.

E.g.,

$$3x^2 + 2x + 3 = 0,$$
$$x^2 + 1 = 0,$$
$$\tfrac{1}{2}x^2 + x\sqrt{2} = 0,$$

are quadratic equations involving one unknown quantity.

The equation $$x^6 + x^3 + 4 = 0$$

is not a quadratic equation in x, but it is one in x^3, for it is the same as

$$(x^3)^2 + (x^3) + 4 = 0.$$

So $$\frac{1}{x^2} + \frac{1}{x} + 2 = 0$$

is, without reduction, a quadratic equation in $\frac{1}{x}$, or x^{-1}, and

$$(a + x^2)^2 + 2(a + x^2) + 3 = 0$$

is a quadratic equation in $a + x^2$, and

$$x^2 + x + 3\sqrt{x^2 + x} = 4$$

is a quadratic equation in $\sqrt{x^2 + x}$.

254. The quadratic equation $ax^2 + bx + c = 0$ is said to be **complete** when neither b nor c is zero; otherwise to be **incomplete.**

The coefficient a cannot be zero, because the equation is to be a quadratic (§ 253).

E.g., $x^2 + 2x - 3 = 0$ is a complete quadratic equation,

but $x^2 - 3 = 0$

and $x^2 + 2x = 0$ are incomplete.

Older English works speak of an equation of the form

$ax^2 + c = 0$ as a **pure quadratic,**

and $ax^2 + bx + c = 0$ as an **affected quadratic.**

255. Solution of incomplete or pure quadratics. An equation in the form $x^2 = a$ can evidently be solved by merely extracting the square root of both members.

If $$x^2 = a,$$

$$\therefore\ x = \pm\sqrt{a}, \text{ two roots.}$$

ILLUSTRATIVE PROBLEMS

1. Solve the equation $3x^2 + 3 = 30$.

If $$3x^2 + 3 = 30,$$

$$\therefore\ x^2 = 9.$$

$$\therefore\ x = +3 \text{ or } -3.$$

2. Solve the equation $(x+2)(x-2) = \frac{1}{4}x^2 - 1$

Multiplying, $$x^2 - 4 = \frac{1}{4}x^2 - 1.$$

$$\therefore\ x^2 = 4.$$

$$\therefore\ x = +2 \text{ or } -2.$$

3. Solve the equation $x^2 + 4 = 0$.

If $$x^2 + 4 = 0,$$

$$x^2 = -4.$$

$$\therefore\ x = \pm\sqrt{-4} = \pm 2\sqrt{-1}.$$

These are *imaginary roots.* See § 242.

EXERCISE CV

Solve the following equations:

1. $x^2 = 625.$
2. $x^2 - 144 = 0.$
3. $x^2 + 289 = 0.$
4. $x^2 + 196 = 0.$
5. $4x^2 - 1600 = 0.$
6. $5x^2 - 1620 = 0.$
7. $5x^2 - 6 = 4x^2 + 3.$
8. $3x^2 + 5 = 5x^2 + 3.$
9. $7x^2 - 5 = 4x^2 + 11.$
10. $5x^2 - 4 = 3x^2 + 46.$
11. $7x^2 - 1 = 5x^2 + 31.$
12. $11x^2 + 7 = 9x^2 + 40.$
13. $12x^2 + 30 = 9x^2 + 50.$
14. $14x^2 - 86 = 10x^2 + 110.$
15. $(x + 5)(x - 4) = x + 5.$
16. $(x + 2)(x + 3) = 5x + 12.$
17. $(x - 4)(x + 7) = 3(x + 10).$
18. $(3x + 2)(2x + 3) = 13(2 + x).$
19. $(4x + 1)(3x - 2) = 5(10 - x).$
20. $\dfrac{x+2}{3} = \dfrac{3}{x-2}.$
21. $\dfrac{x-3}{2} = \dfrac{1-3x}{x-3}.$
22. $\dfrac{x^2}{3} - 4 = \dfrac{x^2}{4} - 1.$
23. $\dfrac{x-2}{x+2} = 4 - \dfrac{x+2}{x-2}.$
24. $\sqrt{3+x} = \dfrac{3}{\sqrt{x-3}}.$
25. $\dfrac{3\sqrt{x-7}}{2} = \dfrac{\frac{1}{2}(x^2 + x - 16)}{\frac{1}{3}\sqrt{x-7}}.$

256. Solution by factoring. One of the best methods of solving the ordinary quadratic equation is by factoring, as already shown in § 101.

ILLUSTRATIVE PROBLEMS

1. Solve the equation

$$x^2 + 16x + 63 = 0.$$

1. This reduces to $(x+9)(x+7)=0$. § 96

2. This is satisfied if either factor is zero, the other remaining finite (§ 123). Hence, either

$$x + 9 = 0, \text{ or } x + 7 = 0.$$

3. $\therefore x = -9, \text{ or } x = -7.$

Check. Substituting these values in the *original* equation (§ 154),

$$81 - 144 + 63 = 0,$$
$$49 - 112 + 63 = 0.$$

2. Solve the equation $2x^2 = 7$.

1. This reduces to $x^2 = \frac{7}{2}$. Ax. 6

2. $\therefore x = \pm\sqrt{\frac{7}{2}} = \pm\frac{1}{2}\sqrt{14}.$ Ax. 9, § 227

That is, it is not worth while to factor as in Ex. 1.

Check. Substituting in the original equation,

$$2 \cdot \tfrac{1}{4} \cdot 14 = 7.$$

3. Solve the equation $6x^2 - 7x + 2 = 0$.

1. This reduces to $(2x-1)(3x-2)=0$. § 97

2. $\therefore 2x - 1 = 0, \text{ or } 3x - 2 = 0.$ § 101

3. $\therefore 2x = 1, \text{ or } 3x = 2,$

and $x = \frac{1}{2}, \text{ or } x = \frac{2}{3}.$

Check. $\frac{6}{4} - \frac{7}{2} + 2 = 0, \quad \frac{8}{3} - \frac{14}{3} + 2 = 0.$

257. Since a quadratic expression has two, and only two, linear factors, *every quadratic equation has two, and only two, roots.*

EXERCISE CVI

Solve the equations:

1. $x+\frac{1}{2}=\frac{1}{2x}.$

2. $\frac{1}{x^2}-\frac{1}{x}=6.$

3. $x^2=x.$

4. $x^2=7-6x.$

5. $9x^2-1=0.$

6. $x^2+2x=0.$

7. $x^2+17x=0.$

8. $x^2+x-2=0.$

9. $x^2=2(12-5x).$

10. $x^2-3x+2=0.$

11. $x(10+x)=-21.$

12. $x^2-2x-15=0.$

13. $x^2+2x-24=0.$

14. $x^2+2x-48=0.$

15. $x^2+2x-80=0.$

16. $x^2-3x-10=0.$

17. $8x-x^2-12=0.$

18. $x^2+3x-10=0.$

19. $x^2+4x-21=0.$

20. $x^2-4x-21=0.$

21. $x^2+5x-14=0.$

22. $x(4-x)+77=0.$

23. $x^2+26x=-120.$

24. $x^2-11x+30=0.$

25. $6x^2+7x+2=0.$

26. $x^2-13x+30=0.$

27. $x^2-15x+56=0.$

28. $x^2-12x-85=0.$

29. $x^2-19x+90=0.$

30. $x^2+19x+18=0.$

31. $x^2+2x-120=0.$

32. $3x^2-10x+3=0.$

33. $x^2-22x+121=0.$

34. $x^2-23x+132=0.$

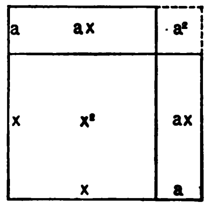
35. $x^2-24x+143=0.$

36. $10x^2+29x=-10.$

258. Solution by completing the square. The addition of an absolute term to two terms so that the trinomial shall be a square is called **completing the square.**

E.g., to complete the square of x^2+2x we must add 1, because x^2+2x+1 is the square of $x+1$.

From the annexed figure it is readily seen that if we have $x^2+ax+ax$, or x^2+2ax, the square on $x+a$ will be completed by adding a^2 in the corner.



259. Since $(x+a)^2=x^2+2ax+a^2$, it is seen that *the quantity which must be added to* $\mathbf{x^2+2ax}$ *to complete the square is the square of half the coefficient of* $\mathbf{x}$.

E.g., to complete the square for x^2+8x, add 16 (the square of $\frac{1}{2}$ of 8), and $x^2+8x+16$ is seen to be the square of $x+4$.

260. By means of completing the square of one member of an equation a solution may be effected, as seen in the following problems.

ILLUSTRATIVE PROBLEMS

1. Solve the equation $x^2-6x=-8$.

Adding the square of half the coefficient of x,

$$x^2-6x+9=1.$$

Extracting the square root of both members,

$$x-3=\pm 1.$$
$$\therefore\ x=3\pm 1,$$
$$=4 \text{ or } 2.$$

2. Solve the equation $x^2+x+1=0$.

Subtracting 1, $\quad x^2+x=-1.$

Adding the square of $\frac{1}{2}$, $\quad x^2+x+\frac{1}{4}=-\frac{3}{4}.$

Extracting the square root, $\quad x+\frac{1}{2}=\pm\frac{1}{2}\sqrt{-3},$

and $\quad x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

261. It therefore appears that *the equation* $x^2 + px + q = 0$ *can be solved by*

1. *Subtracting* q *from each member;* then

2. *Completing the square,* by adding the square of half the coefficient of x (§ 259) to each member; and then

3. *Extracting the square root* of each member and solving the simple equations which are thus obtained.

EXERCISE CVII

Solve the following equations:

1. $\frac{x}{4}+\frac{25}{x}=3.$
2. $1/\left(x+\frac{1}{x}\right)=1.$
3. $x^2+x=132.$
4. $x^2+3x=18.$
5. $x^2+6x=55.$
6. $x^2+4x=60.$
7. $x^2+9x=36.$
8. $x^2-4x=12.$
9. $x^2-6x=27.$
10. $x^2-7x=60.$
11. $x^2-9x=52.$
12. $x^2-2x=-2.$
13. $x^2-9x=-14.$
14. $x^2-6x+2=0.$
15. $x^2-7x+5=0.$
16. $x^2-4x+1=0.$
17. $x^2-6x+4=0.$
18. $x^2+6x+2=0.$
19. $x^2-9x-\frac{1}{5}=0.$
20. $x^2+10x+5=0.$
21. $x^2+8x+18=0.$
22. $x^2+9x+14=0.$
23. $x^2+13x+22=0.$
24. $x^2+17x+60=0.$
25. $x^2-14x+45=0.$
26. $x^2+12x+29=0.$
27. $x^2+10x+23=0.$
28. $x^2+10x+25=0.$
29. $x^2+23x+132=0.$
30. $x^2+23x+130=0.$

262. Solution by formula. Every quadratic equation can be reduced to the form $ax^2+bx+c=0$ (§ 253). Let us solve this general equation.

Dividing by a, and subtracting $\frac{c}{a}$,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$\therefore\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a},\ \text{by adding } \frac{b^2}{4a^2}.$$

Extracting the square root,

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}},$$

whence
$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}},$$

or
$$x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}.$$

Hence, the roots of any quadratic equation which has been reduced to the form $ax^2+bx+c=0$ can be written down at sight.

E.g., the roots of

$$6x^2-13x+6=0 \text{ are } -\frac{-13}{2\cdot 6}\pm\frac{1}{2\cdot 6}\sqrt{(-13)^2-4\cdot 6\cdot 6}$$
$$=\tfrac{13}{12}\pm\tfrac{1}{12}\sqrt{169-144}$$
$$=\tfrac{13}{12}\pm\tfrac{5}{12}=\tfrac{3}{2}\text{ or }\tfrac{2}{3}.$$

Similarly, the roots of

$$\frac{2}{x^2}-\frac{3}{x}+1=0 \text{ are } \frac{1}{x}=-\frac{-3}{2\cdot 2}\pm\frac{1}{2\cdot 2}\sqrt{(-3)^2-4\cdot 2\cdot 1}$$
$$=\tfrac{3}{4}\pm\tfrac{1}{4}\sqrt{9-8}$$
$$=\tfrac{3}{4}\pm\tfrac{1}{4}=1\text{ or }\tfrac{1}{2}.$$
$$\therefore\ x=1\text{ or }2.$$

2. Solve the equation

$$\sqrt{x+1}-4\sqrt{x-4}+5\sqrt{x-7}=0.$$

1. **Isolating the radical $4\sqrt{x-4}$ by adding it to both members. we have :**

$$\sqrt{x+1}+5\sqrt{x-7}=4\sqrt{x-4}.$$

2. Squaring,

$$x+1+25x-175+10\sqrt{x^2-6x-7}=16x-64.$$

3. $$\therefore x-11=-\sqrt{x^2-6x-7}.$$

4. $$\therefore x^2-22x+121=x^2-6x-7.$$

5. $$\therefore x=8.$$

Check. $\sqrt{9}-4\sqrt{4}+5\sqrt{1}=3-8+5=0.$

EXERCISE CIII

Solve the following equations :

1. $\sqrt{x+5}=8.$
2. $\sqrt{x+a}=\frac{1}{2}\sqrt{x}.$
3. $\dfrac{\sqrt{x+5}}{\sqrt{x-2}}=1\frac{1}{3}.$
4. $\dfrac{\sqrt{x}-\sqrt{3}}{\sqrt{x}+\sqrt{3}}=\dfrac{x}{x-3}.$
5. $\dfrac{\sqrt{x}}{\sqrt{x+11}}=\dfrac{5}{6}.$
6. $\dfrac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\dfrac{x}{x-a}.$
7. $\dfrac{\sqrt{x+4}}{\sqrt{x-1}}=1.5.$
8. $\dfrac{\sqrt{x-4}}{\sqrt{x+3}}=\dfrac{\sqrt{x+23}}{\sqrt{x+51}}.$
9. $\dfrac{\sqrt{x+11}}{\sqrt{x+20}}=0.8.$
10. $\dfrac{\sqrt{x+1}}{4}=\dfrac{5}{\sqrt{x+10}}.$
11. $\sqrt{x}+\sqrt{x+7}=7.$
12. $\sqrt{x+2}+\sqrt{x-3}=5.$
13. $\sqrt{2x}+\sqrt{x-4}=6.$
14. $\sqrt{x+2}-\sqrt{x-3}=1.$
15. $\sqrt{x+1}+\sqrt{x-2}=3.$
16. $\sqrt{x-1}+\sqrt{x-6}=5.$

17. $\sqrt{x+5}-\sqrt{x-2}=1.$

18. $\sqrt{x}+3=\sqrt{5}+\sqrt{x+4}.$

19. $\sqrt{x}-2=\sqrt{7}-\sqrt{x-3}.$

20. $\sqrt{x+13}+\sqrt{x+24}=11.$

21. $\sqrt{x-7}+\sqrt{x+9}-8=0.$

22. $\sqrt{3x+4}-\sqrt{x+2}-2=0.$

23. $\sqrt{x+2}+\sqrt{x+3}=\sqrt{2x+5}.$

24. $\sqrt{x-1}+\sqrt{x-4}=\sqrt{2x-1}.$

REVIEW EXERCISE CIV

1. Add $\dfrac{x-a}{x+a}$, $\dfrac{x+a}{x-a}$, and $\dfrac{x^2-5a^2}{x^2-a^2}$.

2. Simplify $\dfrac{(x+y)^2-(x-y)^2}{(x+y)^2+(x-y)^2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$.

3. Reduce to lowest terms the fraction

$$\frac{x^2+p^2x+p^3-p^2}{x^2+(p^2-2p)x-p^3+p^2}.$$

4. Extract the square root of $14+8\sqrt{3}$.

5. Simplify $(14+6\sqrt{5})^{\frac{3}{2}}+(14-6\sqrt{5})^{\frac{3}{2}}$.

6. Simplify $\dfrac{4+\sqrt{2}}{2+\sqrt{2}}+\dfrac{4-\sqrt{2}}{3+\sqrt{2}}+\dfrac{9\sqrt{2}-13}{3-\sqrt{2}}$.

7. Find, to 3 decimal places, the value of $\dfrac{3}{\sqrt{7}}$.

8. Solve the equation $\sqrt{x+5}+1=\sqrt{x-2}+2$.

9. Solve the equation $\dfrac{2}{3x+2}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}$.

267. It will be noticed that the $\pm$ sign has not been placed before the first member when the root has been extracted. The reason for this will appear by considering the following solution:

Given $x^2 = 4$, to find the values of x.

Extracting the square root, and using $\pm$ in both members,

$$\pm x = \pm 2.$$

This means that $\quad x = +2 \text{ or } -2,$

and $\quad -x = +2 \text{ or } -2.$

But we are required to find the values of x (not of $-x$). Hence, multiplying both members of

$$-x = +2 \text{ or } -2$$

by -1, we have $\quad x = -2 \text{ or } +2$, the same as for x above.

268. Hence, *the use of the $\pm$ sign in one member gives the same roots as its use in both members.*

ILLUSTRATIVE PROBLEMS

1. Solve the equation

$$\frac{x+3}{x+5} - \frac{x+1}{x+3} = \frac{3x-5}{3x-7} - \frac{3x-3}{3x-5}.$$

The denominators are such as to suggest adding the fractions in each member separately before clearing of fractions. Then

1. $$\frac{4}{(x+3)(x+5)} = \frac{4}{(3x-5)(3x-7)}.$$

2. Multiplying by $\frac{1}{4}(x+3)(x+5)(3x-5)(3x-7)$,

$$(3x-5)(3x-7) = (x+3)(x+5).$$

3. $$\therefore\ 8x^2 - 44x + 20 = 0,$$

or $$2x^2 - 11x + 5 = 0.$$

4. This is easily factored (§ 256), and

$$(x-5)(2x-1) = 0.$$

5. $$\therefore\ x = 5 \text{ or } \tfrac{1}{2}.$$

Check. For $x = 5$, $\frac{4}{80} = \frac{4}{80}$; for $x = \frac{1}{2}$, $\frac{16}{77} = \frac{16}{77}$.

2. Solve the equation $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}=0.$

Multiplying by $(x-1)(x-2)(x-3)$, we have

1. $$3x^2-12x+11=0.$$

2. This is not so easily factored as in the first problem; hence, applying the formula (§ 264), we have

$$x=-\frac{-12}{2\cdot 3}\pm\frac{1}{2\cdot 3}\sqrt{(-12)^2-4\cdot 3\cdot 11}$$

$$=2\pm\tfrac{1}{3}\sqrt{3}.$$

Check. $$\frac{1}{1\pm\frac{1}{3}\sqrt{3}}+\frac{1}{\pm\frac{1}{3}\sqrt{3}}+\frac{1}{-1\pm\frac{1}{3}\sqrt{3}}$$

$$=\frac{1\mp\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}}\pm\sqrt{3}+\frac{-1\mp\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}}$$

$$=\tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{3}\pm\sqrt{3}-\tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{3}=0.$$

3. Solve the equation $x^2+2x=0.$

This factors into $x(x+2)=0$, whence $x=0$ or -2.

And, in general, if x is a factor of every term of an equation, $x=0$ is one root.

4. Solve the equation $2x^2-2x=5.$

1. $x^2-x=\frac{5}{2}.$ Ax. 7
2. $x^2-x+\frac{1}{4}=\frac{11}{4}.$ Ax. 2
3. $x-\frac{1}{2}=\pm\frac{1}{2}\sqrt{11}.$ Ax. 9
4. $x=\frac{1}{2}(1\pm\sqrt{11}).$ Ax. 2

Check. $(6\pm\sqrt{11})-(1\pm\sqrt{11})=5.$

It is often possible, in cases of this kind, to avoid fractions by the exercise of a little forethought. This equation may be written

1′. $4x^2-4x=10.$

2′. $\therefore\ (2x)^2-2(2x)+1=11$, a quadratic in $2x$.

3′. $\therefore\ 2x-1=\pm\sqrt{11}.$

4′. $\therefore\ 2x=1\pm\sqrt{11}.$

5′. $\therefore\ x=\frac{1}{2}(1\pm\sqrt{11}).$

III. THE MEANING OF THE FRACTIONAL EXPONENT

201. We have now found the meaning of

1. The positive integral exponent greater than 1, the primitive meaning of exponent;
2. The unit exponent;
3. The zero exponent;
4. The negative integral exponent.

202. It remains to find the meaning which should attach to the **fractional exponent.**

The expression a^4 means *aaaa,*
and if the exponent is half as large,

a^2 or aa is the square root of a^4,

and if the exponent is half as large,

a^1 or a is the square root of a^2.

$\therefore$ if an exponent half as large indicates a square root,

$a^{\frac{1}{2}}$ should mean the square root of a.

Hence, $a^{\frac{1}{2}}$ is defined to mean the square root of a, and, in general, $a^{\frac{1}{n}}$ is defined to mean the nth root of a.

203. The reason for this is also seen from the fact that

$\because\ a^m \cdot a^m \cdots$ to n factors $= a^{mn}$,

$\therefore\ a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots$ " " should equal $a^{n\left(\frac{1}{n}\right)}$ or $a^{\frac{n}{n}}$ or a.

$\therefore\ a^{\frac{1}{n}}$ should be defined to mean the nth root of a.

204. And since $a^{mn} = (a^m)^n$, so $a^{\frac{p}{q}}$ should be defined to be identical with $(a^{\frac{1}{q}})^p$.

Hence, *we define* $a^{\frac{p}{q}}$ *to mean the* p*th power of the* q*th root of* a, *and* $a^{-\frac{p}{q}}$ *to mean the reciprocal of* $a^{\frac{p}{q}}$.

ILLUSTRATIVE PROBLEMS

1. Find the absolute value of $343^{-\frac{2}{3}}$.

$$343^{-\frac{2}{3}} = \frac{1}{(343^{\frac{1}{3}})^2} = \frac{1}{7^2} = \frac{1}{49}.$$

2. Write in integral form, with negative or fractional exponents $1 \div \sqrt[7]{x^6}$, or, as in § 206, $1 \div (\sqrt[7]{x})^6$.

$$1 \div \sqrt[7]{x^6} = 1 \div x^{\frac{6}{7}} = x^{-\frac{6}{7}}.$$

3. Write without negative or fractional exponents $a^{-\frac{1}{5}}$.

$$a^{-\frac{1}{5}} = \frac{1}{a^{\frac{1}{5}}} = \frac{1}{\sqrt[5]{a}}.$$

EXERCISE LXXXVI

Find the absolute value of the expressions in Exs. 1–15.

1. $4^{\frac{1}{2}}$. 2. $9^{\frac{1}{2}}$. 3. $8^{\frac{1}{3}}$. 4. $32^{\frac{1}{5}}$. 5. $81^{\frac{1}{4}}$.

6. $25^{\frac{3}{2}}$. 7. $125^{\frac{1}{3}}$. 8. $32^{\frac{3}{5}}$. 9. $64^{\frac{5}{6}}$. 10. $625^{\frac{3}{4}}$.

11. $16^{-\frac{1}{2}}$. 12. $36^{-\frac{3}{2}}$. 13. $27^{\frac{2}{3}}$. 14. $16^{-\frac{3}{4}}$. 15. $32^{\frac{3}{5}}$.

Write in integral form, with negative or fractional exponents, the expressions in Exs. 16–23.

16. $\sqrt{1 \div a^3}$.

17. $\sqrt[3]{1 \div a^2}$.

18. $\sqrt{1 \div b^3\sqrt{c}}$.

19. $\sqrt[3]{1 \div (a+b)^2}$.

20. $\dfrac{a}{\sqrt[4]{b}} + \dfrac{a+b}{\sqrt[3]{a-b}} - \dfrac{\frac{1}{a}}{\sqrt{a}+\sqrt[3]{a}} - \dfrac{1}{a^2}$.

21. $a\sqrt[3]{b} + b\sqrt[3]{a} + \sqrt[3]{a+b} - \sqrt[n]{a-b}$.

22. $\dfrac{1}{a\sqrt{a}} + \dfrac{1}{b\sqrt{b}} - \dfrac{\sqrt{a^3}+\sqrt{b^3}}{\sqrt{a^3}} + \dfrac{1}{a^3} - \dfrac{1}{b^3}$.

23. $a^2 \div (\sqrt{a} + \sqrt{1+a}) \div (a^3 + \sqrt[3]{a^2}) \div \sqrt{a^3}$.

Write the following using the old form of radical sign ($\surd$) and the common fraction:

24. $z^{\frac{1}{6}}$.	25. $x^{\frac{m}{n}}$.	26. $a^{-\frac{1}{5}}$.	27. $x^{\frac{m+1}{2}}$.
28. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.	29. $a^{\frac{2}{3}}b^{\frac{1}{3}}$.	30. $x^{\frac{1}{2}}y^{\frac{1}{3}}$.	31. $a^{-\frac{2}{3}}b^{-\frac{2}{3}}$.
32. $x^{\frac{7}{5}}y^{-\frac{7}{8}}$.	33. $x^{\frac{m+n}{m-n}}y^{\frac{m-n}{m+n}}$.	34. $m^{\frac{2}{4}}n^{\frac{4}{5}}$.	35. $x^{-1}y^{\frac{1}{2}}$.

205. It has been proved that $(abc\cdots)^m = a^m b^m c^m \cdots$. It is equally true that $(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} c^{\frac{1}{n}} \cdots$.

For let $\qquad x = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Then $\qquad x^n = (a^{\frac{1}{n}} b^{\frac{1}{n}})^n = ab$. § 178

$$\therefore\ x = (ab)^{\frac{1}{n}}.$$

$$\therefore\ (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}. \qquad \text{Ax. 1}$$

The same reasoning holds for $(abc\cdots)^{\frac{1}{n}}$.

Similarly $\qquad \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$.

206. It has been proved that $(a^m)^n = a^{mn} = (a^n)^m$. It is equally true that $(a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$.

For $\qquad (aaa \cdots \text{to } m \text{ factors})^{\frac{1}{n}} = a^{\frac{1}{n}} a^{\frac{1}{n}} a^{\frac{1}{n}} \cdots \text{to } m \text{ factors}$. § 205

I.e., $\qquad (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m$.

But $\qquad (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$. Def. § 204

Hence, $a^{\frac{m}{n}}$ may be considered either as the mth power of the nth root of a or as the nth root of the mth power of a.

But this must be understood to apply only to the *absolute values* of the roots.

E.g., $\qquad (4^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \pm 4$,

but $\qquad (4^{\frac{1}{2}})^2 = (\pm 2)^2 = +4$.

207. Fractional exponents are subject to the laws of common fractions, although they are only fractions in *form*. For example,

$$a^{\frac{m}{n}} = a^{\frac{pm}{pn}}.$$

For let $x = a^{\frac{m}{n}}$.

Then $x^n = a^m$, by raising to nth power,

and $\therefore\ x^{pn} = a^{pm}$, " " " pth "

$\therefore\ x = a^{\frac{pm}{pn}}$.

Therefore $a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$. Ax. 1

Hence, *both terms of a fractional exponent can be multiplied or divided by the same number without altering the value of the expression.*

The student should understand clearly that this is true *not because the exponent is a fraction.* The exponent is merely an expression in the *form* of a fraction, and hence a proof like that of § 118 has no application to this case. The laws of fractions apply to fractional exponents only as they are proved to do so.

208. It has been proved that $(a^m)^n = a^{mn}$. It is equally true that

$$(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}} = (a^{\frac{1}{n}})^{\frac{1}{m}}.$$

For let $x = (a^{\frac{1}{m}})^{\frac{1}{n}}$.

Then $x^n = a^{\frac{1}{m}}$,

and $x^{mn} = a$.

Therefore $x = a^{\frac{1}{mn}}$.

Similarly if $x = (a^{\frac{1}{n}})^{\frac{1}{m}}$,

it follows that $x = a^{\frac{1}{mn}}$.

$$\therefore\ x = (a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}.$$

IV. THE LAWS FOR NEGATIVE AND FRACTIONAL EXPONENTS

209. Having now found the meaning of the negative and the fractional exponents, and having proved certain laws concerning them, it remains to prove that the three fundamental laws of exponents,

$$a^m \cdot a^n = a^{m+n},$$
$$a^m : a^n = a^{m-n},$$
$$(a^m)^n = a^{mn},$$

are true, *if* m *and* n *are fractional, negative, or both fractional and negative.*

210. To prove that $a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$ or $a^{\frac{ps+qr}{qs}}$.

We know from § 207 that

$$\begin{aligned} a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} &= a^{\frac{ps}{qs}} \cdot a^{\frac{qr}{qs}} & \\ &= (a^{ps} \cdot a^{qr})^{\frac{1}{qs}} & \S\, 205 \\ &= (a^{ps+qr})^{\frac{1}{qs}} & \S\, 60 \\ &= a^{\frac{ps+qr}{qs}} & \S\, 204 \end{aligned}$$

This shows that a case like $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ can be easily handled by fractional exponents, thus:

$$a^{\frac{2}{3}} \cdot a^{\frac{4}{5}} = a^{\frac{2}{3}+\frac{4}{5}} = a^{\frac{22}{15}}.$$

To see that $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ equals the 15th root of a^{22} is not so easy by the help of the old symbols alone.

211. To prove that $a^{\frac{p}{q}} : a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}$.

The proof is evidently identical with that just given, except that the sign of division replaces that of multiplication in the first member, and the sign of subtraction that of addition in the second member.

212. These laws (§§ 205–211) are of great importance in simplifying algebraic expressions.

ILLUSTRATIVE PROBLEMS

1. Simplify $(4\,a^2b^6)^{\frac{1}{2}}$.

By § 205,
$$(4\,a^2b^6)^{\frac{1}{2}} = 4^{\frac{1}{2}}\,a^{\frac{2}{2}}b^{\frac{6}{2}}$$
$$= 2\,ab^3. \qquad \text{§ 210}$$

2. Simplify $(2\,a^{\frac{1}{2}}b^{\frac{1}{4}})^4$.

By § 178,
$$(2\,a^{\frac{1}{2}}b^{\frac{1}{4}})^4 = 2^4\,a^{\frac{4}{2}}b^{\frac{4}{4}}$$
$$= 16\,a^2b. \qquad \text{§ 207}$$

3. Simplify $\sqrt[3]{a^6b^9}$.

By definition of fractional exponent,
$$\sqrt[3]{a^6b^9} = (a^6b^9)^{\frac{1}{3}}$$
$$= a^2b^3. \qquad \text{§ 205}$$

The same result may be found by § 191.

4. Simplify $\sqrt{a} \cdot \sqrt[3]{a^2}$.

$$\sqrt{a} \cdot \sqrt[3]{a^2} = a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}$$
$$= a^{\frac{1}{2}+\frac{2}{3}}$$
$$= a^{\frac{7}{6}} = a \cdot a^{\frac{1}{6}} \text{ or } a\sqrt[6]{a}.$$

5. Simplify $\dfrac{\sqrt{a^3}}{\sqrt[3]{a^2}}$.

$$\frac{\sqrt{a^3}}{\sqrt[3]{a^2}} = a^{\frac{3}{2}-\frac{2}{3}} = a^{\frac{5}{6}} \text{ or } \sqrt[6]{a^5}.$$

6. Simplify $a^{\frac{2}{3}}b^{\frac{3}{4}} \cdot a^{\frac{1}{6}}b^{\frac{3}{8}} \cdot b^{\frac{1}{2}} \cdot (a^{\frac{1}{12}})^2 \cdot b^{\frac{1}{8}} \cdot b^{\frac{1}{4}}$.

By §§ 206, 207, 210, this equals
$$a^{\frac{2}{3}+\frac{1}{6}+\frac{1}{6}}b^{\frac{3}{4}+\frac{3}{8}+\frac{1}{2}+\frac{1}{8}+\frac{1}{4}},$$
which equals ab^2.

EXERCISE LXXXVII

Simplify the following expressions:

1. $x^{\frac{1}{4}} \cdot x^{\frac{1}{5}}$.

2. $x^{\frac{1}{3}} \cdot x^{\frac{1}{2}} \cdot x$.

3. $(x^7y^{21}z^{35})^7$.

4. $(3\,a^{\frac{1}{2}}b^{\frac{1}{2}}x^{\frac{1}{4}}y^{\frac{1}{4}})^4$.

5. $(a^{\frac{1}{6}}b^{\frac{1}{4}}c^{\frac{1}{3}}d)^{12}$.

6. $(81\,a^{12}x^{20}y^{28})^{\frac{1}{4}}$.

7. $\sqrt[3]{a^6b^{12}x^9y^{15}}$.

8. $(w^{20}x^{40}y^{60}z^{80})^{0.1}$.

9. $(2\,a^{\frac{1}{7}}b^{\frac{1}{7}}cd^2)^7$.

10. $(a^{10}b^{20}c^{30}d^{40})^{\frac{1}{10}}$.

11. $(27\,a^9b^{12}c^3)^{\frac{1}{3}}$.

12. $(-\,a^{\frac{1}{3}}b^{\frac{1}{6}}x^{\frac{1}{4}}y)^{12}$.

13. $\sqrt[6]{a^{12}b^{24}c^{30}d^{42}}$.

14. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}}q^{\frac{1}{7}}r)^{140}$.

15. $(p^{\frac{1}{2}}q^{\frac{1}{3}}r^{\frac{1}{4}}s^{\frac{1}{5}})^{60}$.

16. $\sqrt[3]{-\,m^6n^9x^{15}y^{21}}$.

17. $(64\,a^{12}b^{21}c^6)^{\frac{1}{3}}$.

18. $(125\,a^3b^6c^9d^{12})^{\frac{1}{3}}$.

19. $(a^{12}b^{18}c^{30}d^{36})^{\frac{1}{6}}$.

20. $(343\,m^{12}n^{15}p^{18})^{\frac{1}{3}}$.

21. $(32\,a^{20}b^{30}c^{40})^{\frac{1}{5}}$.

22. $\sqrt[5]{-\,32\,a^{10}b^{20}c^{30}d^{40}}$.

23. $\sqrt[3]{x^2yz} : \sqrt{xyz}$.

24. $x^{\frac{1}{a}} \cdot x^{\frac{3}{a}} : x^{\frac{2}{a}} \cdot x^{\frac{a-2}{a}}$.

25. $\sqrt[4]{x^3yz} : \sqrt[5]{x^4y^3z^2}$.

26. $\sqrt[4]{a^3} \cdot \sqrt{a} \cdot \sqrt[3]{a^2}$.

27. $a^{\frac{1}{2}}b^{\frac{1}{3}} \cdot a^{\frac{1}{3}}b^{\frac{1}{2}} \cdot a^{\frac{1}{6}}b^{\frac{1}{6}}$.

28. $\sqrt[5]{a^2bx^2y} : \sqrt[3]{a^2bx^2y}$.

29. $p \cdot p^{\frac{1}{2}} \cdot p^{\frac{1}{3}} \cdot p^{\frac{1}{4}} : p^{\frac{1}{12}}$.

30. $\sqrt[3]{ab} \cdot \sqrt[4]{a^3b} \cdot \sqrt[6]{a^5b^4}$.

31. $a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}} \cdot a^{\frac{2}{3}}b^{\frac{3}{4}}c^{\frac{2}{5}} \cdot a^2b^3c^{\frac{3}{5}}$.

32. $\sqrt[7]{-\,p^2q^3x^5y^6} \cdot \sqrt[6]{p^3q^4x^3y^4} : \sqrt[5]{pq^2x^3y^4}$.

33. $\sqrt[3]{x^2y^2z \cdot x^4y^4z \cdot xyz} : \sqrt[4]{x^3y^3z^3 \cdot x^2y^2z^2}$.

213. To prove that $a^m \cdot a^{-n} = a^{m+(-n)}$, or a^{m-n}.

$$a^m \cdot a^{-n} \text{ means } a^m \cdot \frac{1}{a^n}$$
$$= \frac{a^m}{a^n} \qquad \S\ 130$$
$$= a^{m-n}. \qquad \S\ 72$$

214. In the case of $a^m : a^{-n} = a^{m-(-n)} = a^{m+n}$, the proof is evidently identical with that just given, except that the sign of division replaces that of multiplication, and the sign of subtraction that of addition.

215. To prove that $a^{-m} \cdot a^{-n} = a^{-m+(-n)} = a^{-m-n}$.

By definition
$$a^{-m} \cdot a^{-n} = \frac{1}{a^m} \cdot \frac{1}{a^n} \qquad \S\ 200$$
$$= \frac{1}{a^m a^n}$$
$$= \frac{1}{a^{m+n}}$$
$$= a^{-m-n}. \qquad \S\ 200$$

216. It has now been shown that
$$a^m \cdot a^n = a^{m+n}$$
and
$$a^m : a^n = a^{m-n},$$
whether m and n are fractional, negative, or both fractional and negative.

ILLUSTRATIVE PROBLEM

Simplify $\dfrac{1}{\sqrt[5]{a^4}} : \dfrac{1}{\sqrt[4]{a^3}}$.

Here we have
$$a^{-\frac{4}{5}} : a^{-\frac{3}{4}} = a^{\frac{-16+15}{20}} = a^{-\frac{1}{20}},$$
or the 20th root of $\dfrac{1}{a}$, a result not so easily reached by the older notation.

EXERCISE LXXXVIII

Simplify the following expressions:

1. $\dfrac{a}{\sqrt[3]{a}} \cdot \dfrac{\sqrt{a}}{a^{\frac{1}{6}}}$.

2. $\dfrac{1}{\sqrt[3]{a}} : \dfrac{1}{\sqrt[4]{a^3}}$.

3. $\dfrac{1}{\sqrt[5]{x^4}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[15]{x^{13}}}$.

4. $\dfrac{1}{\sqrt[3]{a^2b}} \cdot \dfrac{1}{\sqrt{ab^3}}$.

5. $\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{a^3b}} : \dfrac{\sqrt[6]{a^3b^2}}{\sqrt[12]{a^7b^9}}$.

6. $\dfrac{a^2}{a^{\frac{1}{3}}} \cdot \dfrac{a^{\frac{1}{5}}}{a^{\frac{1}{15}}} \quad \sqrt[5]{a^4}$.

7. $\dfrac{a^2}{\sqrt[3]{a^2}} : \dfrac{\sqrt{a}}{a^3} : \dfrac{\sqrt[3]{a}}{\sqrt{a}}$.

8. $a^{-2} \cdot a^{\frac{2}{3}} \cdot a^3 \cdot \sqrt[3]{a}$.

217. It remains to prove that the law that $(a^m)^n = a^{mn}$ is true if m and n are fractional, negative, or both fractional and negative. Since the proofs are so nearly like those already given, only a single case need be considered.

218. To prove that $(a^m)^{-n} = a^{-mn}$.

By definition

$$(a^m)^{-n} = \frac{1}{(a^m)^n}, \qquad \text{§ 200}$$

$$= \frac{1}{a^{mn}}, \qquad \text{§ 177}$$

$$= a^{-mn}. \qquad \text{§ 200}$$

ILLUSTRATIVE PROBLEMS

1. Simplify $$\sqrt[3]{\left(\frac{1}{1 \div \sqrt[4]{a^3}}\right)^4}.$$

This expression, thus written in the older style, does not strike the eye as simple; but since $1 \div \sqrt[4]{a^3}$ may be written $a^{-\frac{3}{4}}$, the expression reduces to $(a^{\frac{3}{4}})^{\frac{4}{3}}$, which equals a.

2. Simplify $\quad x \cdot \sqrt[r^2-q^2]{\left(\dfrac{1}{\sqrt[q]{x^r}} \div \dfrac{1}{\sqrt[r]{x^q}}\right)^{qr}}.$

Writing this with fractional and negative exponents, we have

$$x \cdot \left(x^{-\frac{r}{q}+\frac{q}{r}}\right)^{\frac{qr}{r^2-q^2}} = x \cdot x^{-\frac{r^2-q^2}{qr} \cdot \frac{qr}{r^2-q^2}} = x \cdot x^{-1} = x^0 = 1.$$

To simplify this without the assistance of negative and fractional exponents would be more difficult.

EXERCISE LXXXIX

Simplify the following expressions:

1. $(a^{\frac{3}{4}}b^{\frac{5}{6}})^{\frac{2}{3}}$.
2. $a^{\frac{2}{3}} \cdot a^{\frac{3}{4}}$.
3. $[(-a^3)^2]^5$.
4. $\sqrt[8]{x^7y^6z^5}$.
5. $(x^{-1}+y^{-1})^3$.
6. $\sqrt[p]{a^{-q}b^{2p}}$.
7. $[(x^{\frac{2}{3}}y^{-\frac{5}{9}}z^{\frac{4}{5}})^{-\frac{7}{8}}]^{\frac{1}{4}}$.
8. $\sqrt[m]{a^{2m}b^{3m^2}}$.
9. $(2x^{-2} \div y^{-2})^{-3}$.
10. $\sqrt[m+n]{a^{m^2-n^2}}$.
11. $\sqrt[13]{(\sqrt[3]{x^2} \cdot \sqrt{x^4})^{14}}$.
12. $a^{-\frac{p}{q}}b^{-x}\sqrt[3]{cd^2}$.
13. $\{[(x^{-2})^{-2}]^{-2}\}^{-2}$.
14. $x^{-x} \cdot (-x^x)$.
15. $(x^{m-n})^{m+n} \cdot x^{n^2} : x^{m^2}$.
16. $\sqrt[m]{a^{-m}b^{-2m}c^{-m^2}}$.
17. $\sqrt[3]{64\,[(x-y)^{-6}]^{\frac{1}{2}}}$.
18. $5\sqrt[3]{x^2y} \cdot 2x^{\frac{3}{4}}y^2$.
19. $\{[(x^2-y^{-2})^{-1}]^3\}^{-2}$.
20. $ax^{-m}y^{-n} \cdot bx^ny^m$.
21. $[(a^{m+n})^{m-n} \cdot (a^{n^3})^{\frac{1}{n}}]^{\frac{1}{m^2}}$.
22. $3a^{-\frac{2}{3}} \cdot 4a^{-\frac{5}{6}} \cdot 2a^{\frac{3}{4}}$.
23. $[(a^{-p})^q]^{-\frac{1}{p}} : [(a^q)^{-r}]^{-\frac{1}{r}}$.
24. $3a^{-2}b^{-3}c^4 \cdot 4a^{-3}b^5c^{-4}$.
25. $-a^{-4}b^4c^5d^{-5} \cdot -a^5b^{-5}c^4d^{-4}$.

EXERCISE LXXXVIII

Simplify the following expressions:

1. $\dfrac{a}{\sqrt[3]{a}} \cdot \dfrac{\sqrt{a}}{a^{\frac{1}{6}}}$.

2. $\dfrac{1}{\sqrt[3]{a}} : \dfrac{1}{\sqrt[4]{a^3}}$.

3. $\dfrac{1}{\sqrt[5]{x^4}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[15]{x^{13}}}$.

4. $\dfrac{1}{\sqrt[3]{a^2b}} \cdot \dfrac{1}{\sqrt{ab^3}}$.

5. $\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{a^3b}} : \dfrac{\sqrt[6]{a^3b^2}}{\sqrt[12]{a^7b^9}}$.

6. $\dfrac{a^2}{a^{\frac{1}{3}}} \cdot \dfrac{a^{\frac{1}{5}}}{a^{\frac{1}{15}}} \quad \sqrt[5]{a^4}$.

7. $\dfrac{a^2}{\sqrt[3]{a^2}} : \dfrac{\sqrt{a}}{a^3} : \dfrac{\sqrt[3]{a}}{\sqrt{a}}$.

8. $a^{-2} \cdot a^{\frac{2}{3}} \cdot a^3 \cdot \sqrt[3]{a}$.

217. It remains to prove that the law that $(a^m)^n = a^{mn}$ is true if m and n are fractional, negative, or both fractional and negative. Since the proofs are so nearly like those already given, only a single case need be considered.

218. To prove that $(a^m)^{-n} = a^{-mn}$.

By definition $\quad (a^m)^{-n} = \dfrac{1}{(a^m)^n}, \quad$ § 200

$= \dfrac{1}{a^{mn}}, \quad$ § 177

$= a^{-mn}. \quad$ § 200

ILLUSTRATIVE PROBLEMS

1. Simplify $\quad \sqrt[3]{\left(\dfrac{1}{1 \div \sqrt[4]{a^3}}\right)^4}$.

This expression, thus written in the older style, does not strike the eye as simple; but since $1 \div \sqrt[4]{a^3}$ may be written $a^{-\frac{3}{4}}$, the expression reduces to $(a^{\frac{3}{4}})^{\frac{4}{3}}$, which equals a.

2. Simplify $$x\cdot\sqrt[r^2-q^2]{\left(\frac{1}{\sqrt[q]{x^r}}\div\frac{1}{\sqrt[r]{x^q}}\right)^{qr}}.$$

Writing this with fractional and negative exponents, we have

$$x\cdot\left(x^{-\frac{r}{q}+\frac{q}{r}}\right)^{\frac{qr}{r^2-q^2}}=x\cdot x^{-\frac{r^2-q^2}{qr}\cdot\frac{qr}{r^2-q^2}}=x\cdot x^{-1}=x^0=1.$$

To simplify this without the assistance of negative and fractional exponents would be more difficult.

EXERCISE LXXXIX

Simplify the following expressions:

1. $(a^{\frac{3}{4}}b^{\frac{5}{6}})^{\frac{2}{3}}$.
2. $a^{\frac{2}{3}}\cdot a^{\frac{2}{3}}$.
3. $[(-a^3)^2]^5$.
4. $\sqrt[3]{x^7y^6z^5}$.
5. $(x^{-1}+y^{-1})^3$.
6. $\sqrt[p]{a^{-q}b^{2p}}$.
7. $[(x^{\frac{2}{3}}y^{-\frac{5}{9}}z^{\frac{4}{5}})^{-\frac{7}{8}}]^{\frac{1}{4}}$.
8. $\sqrt[m]{a^{2m}b^{3m^2}}$.
9. $(2x^{-2}\div y^{-2})^{-3}$.
10. $\sqrt[m+n]{a^{m^2-n^2}}$.
11. $\sqrt[13]{(\sqrt[3]{x^2}\cdot\sqrt{x^4})^{14}}$.
12. $a^{-\frac{p}{q}}b^{-x}\sqrt[3]{cd^2}$.
13. $\{[(x^{-2})^{-2}]^{-2}\}^{-2}$.
14. $x^{-x}\cdot(-x^x)$.
15. $(x^{m-n})^{m+n}\cdot x^{n^2}:x^{m^2}$.
16. $\sqrt[m]{a^{-m}b^{-2m}c^{-m^2}}$.
17. $\sqrt[3]{64\,[(x-y)^{-6}]^{\frac{1}{2}}}$.
18. $5\sqrt[3]{x^2y}\cdot 2\,x^{\frac{3}{4}}y^2$.
19. $\{[(x^2-y^{-2})^{-1}]^3\}^{-2}$.
20. $ax^{-m}y^{-n}\cdot bx^ny^m$.
21. $[(a^{m+n})^{m-n}\cdot(a^{n^3})^{\frac{1}{n}}]^{\frac{1}{m^2}}$.
22. $3\,a^{-\frac{2}{3}}\cdot 4\,a^{-\frac{5}{6}}\cdot 2\,a^{\frac{3}{4}}$.
23. $[(a^{-p})^q]^{-\frac{1}{p}}:[(a^q)^{-r}]^{-\frac{1}{r}}$.
24. $3\,a^{-2}b^{-3}c^4\cdot 4\,a^{-3}b^5c^{-4}$.
25. $-a^{-4}b^4c^5d^{-5}\cdot -a^5b^{-5}c^4d^{-4}$.

V. PROBLEMS INVOLVING FRACTIONAL AND NEGATIVE EXPONENTS

219. It has now been proved that we can operate with expressions involving negative or fractional exponents just as if these exponents were positive integers. Exercises involving such exponents will be given on pp. 224, 225.

The student should see the distinct advantage in using the fractional exponent instead of the old form of radical sign, except in cases like the expression of a single root, *and in using the negative exponent,* except in cases like the expression of a simple fraction. This has been shown on pp. 220, 221, but it is worth while to consider the matter further, that the student may become entirely familiar with the use of the modern symbols.

E.g., while it is easier to write $\sqrt{a}$ than $a^{\frac{1}{2}}$, and $\frac{1}{a}$ than a^{-1}, because we are more accustomed to the forms $\sqrt{a}$ and $\frac{1}{a}$, it is much easier to see that

$$(x^{-\frac{2}{3}})^{-\frac{3}{4}} = x^{\frac{1}{2}},$$

than to see that the equivalent expression

$$\frac{1}{\sqrt[4]{(1 \div \sqrt[3]{x^2})^3}} = \sqrt{x}.$$

To take another example, it is doubtful if students would readily grasp the significance of the form $a^3 + 2a^2\sqrt[6]{a} + a\sqrt[3]{a}$; but when written $a^{\frac{6}{2}} + 2a^{\frac{13}{6}} + a^{\frac{4}{3}}$, it is seen to be the square of $a^{\frac{3}{2}} + a^{\frac{2}{3}}$.

In the case of polynomials the value of the negative and fractional exponents is also quite as evident as in that of monomials.

E.g., the eye more readily takes in the operation suggested by the symbols

$$(x^{\frac{5}{6}} - y^{\frac{1}{6}})(2x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}),$$

than the same operation expressed by the symbols

$$(\sqrt[6]{x^5} - \sqrt[6]{y})\left(\frac{2}{\sqrt[3]{x^2}} + \frac{3}{\sqrt[3]{x}\sqrt[3]{y}} + \frac{1}{\sqrt[3]{y^2}}\right).$$

ILLUSTRATIVE PROBLEMS

1. Remove the parentheses from $(x^{-1} \div y^{-1})^{-2}$, expressing the result with positive exponents.

$$(x^{-1} \div y^{-1})^{-2} = x^2 \div y^2. \qquad \text{§ 218}$$

2. Multiply $x^{-2} + x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Since we can multiply as if the exponents were positive, we have the following:

$$\begin{array}{lllll}
x^{-2} & + x^{-1} & + 1 & & \\
x^{-2} & - x^{-1} & + 1 & & \\
\hline
x^{-4} & + x^{-3} & + x^{-2} & & \\
 & - x^{-3} & - x^{-2} & - x^{-1} & \\
 & & + x^{-2} & + x^{-1} & + 1 \\
\hline
x^{-4} & & + x^{-2} & & + 1
\end{array}$$

Detached coefficients may be used in practice.

3. Divide $x^{-3} + 3x^{-2} + 3x^{-1} + 1$ by $x^{-1} + 1$.

Since we can divide as if the exponents were positive, we have the following:

$$\text{Quotient} = x^{-2} + 2x^{-1} + 1$$

$$\begin{array}{r|llll}
x^{-1} + 1 & x^{-3} & + 3x^{-2} & + 3x^{-1} & + 1 \\
 & x^{-3} & + x^{-2} & & \\
\hline
 & & 2x^{-2} & + 3x^{-1} & \\
 & & 2x^{-2} & + 2x^{-1} & \\
\hline
 & & & x^{-1} & + 1 \\
 & & & x^{-1} & + 1 \\
\hline
\end{array}$$

Detached coefficients may be used in practice.

It is evident that we may check the work by arbitrary values as in the case of positive integral exponents.

Thus Ex. 3, let $x = 1$; then

$$(1 + 3 + 3 + 1) \div (1 + 1) = 1 + 2 + 1,$$
$$8 \div 2 = 4.$$

EXERCISE XC

Perform the operations indicated in the following exercises:

1. $\dfrac{a^{\frac{4}{5}}b^{\frac{3}{4}}}{c^{\frac{2}{3}}} : \dfrac{a^{\frac{3}{4}}b^{\frac{2}{3}}}{c^{\frac{5}{6}}}$.

2. $(x^{\frac{m}{n}} + x^{\frac{n}{m}})^2$.

3. $4\,ab^{\frac{2}{3}}c^{\frac{1}{4}} : 2\,b^{\frac{4}{5}}c^{\frac{7}{8}}$.

4. $(x^{-\frac{1}{n}} - 1)^2$.

5. $\sqrt{a^{-2} + 2 + a^2}$.

6. $(a^{\frac{1}{3}} + b^{\frac{2}{3}})^2$.

7. $3\,a^{\frac{1}{3}}b^{\frac{1}{4}} : 1.5\,a^{\frac{1}{6}}b^{\frac{1}{8}}$.

8. $(a^{\frac{1}{3}} - b^{-\frac{1}{3}})^3$.

9. $a^{\frac{1}{2}}b^{\frac{5}{7}} : a^{\frac{1}{4}}b^{\frac{4}{7}} \cdot a^{\frac{3}{4}}b^{\frac{6}{7}}$.

10. $(a^{-b} - b^{-a})^3$.

11. $(x + y)(x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

12. $(x^{\frac{2}{3}} + y^{-\frac{1}{3}})^2$.

13. $\sqrt{9\,x^{\frac{4}{7}} + 6\,x^{\frac{2}{7}} + 1}$.

14. $(a^{\frac{1}{4}}b^{\frac{1}{2}}c + 1)^2$.

15. $\sqrt{4\,x^{\frac{2}{5}} + 4\,x^{\frac{1}{5}} + 1}$.

16. $(a^{-1} + a^{-2})^{-2}$.

17. $a^{\frac{2}{3}}b^{\frac{3}{7}} \cdot a^{\frac{5}{6}}b^{\frac{1}{14}} : a^{\frac{2}{3}}b^{\frac{1}{7}}$.

18. $\dfrac{a^{\frac{2}{3}}b}{ab^{\frac{2}{3}}} \cdot a^{\frac{1}{3}} \cdot \dfrac{ab^{\frac{2}{3}}}{b^{\frac{1}{3}}}$.

19. $(a^{-1} + a^{-2} + a^{-3})^2$.

20. $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)^2$.

21. $\sqrt{a^{\frac{2}{3}} + 2\,a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$.

22. $2\,a^{\frac{1}{2}}b^{\frac{1}{4}} \cdot 3\,a^{\frac{1}{4}}b^{\frac{1}{2}}$.

23. $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} - y^{\frac{2}{3}})$.

24. $\sqrt{x^{-4} + x^4 - 2}$.

25. $(a^2 - a^{-2})(a^2 + a^{-2})$.

26. $3\,a^{\frac{2}{3}}x^{\frac{1}{4}} \cdot 6\,a^{\frac{1}{6}}x^{\frac{1}{2}}$.

27. $a^{\frac{2}{3}}b^{\frac{1}{3}}c \cdot a^{\frac{1}{6}}b^{\frac{1}{6}}c^{\frac{5}{6}} \cdot a^{\frac{5}{6}}b^{\frac{1}{3}}c^{\frac{1}{3}}$

28. $(a^2b^{-2}c^{\frac{1}{2}}d^{-\frac{1}{2}} + 1)^2$.

29. $\sqrt[3]{x^{\frac{3}{5}} + 3\,x^{\frac{2}{5}} + 3\,x^{\frac{1}{5}} + 1}$.

30. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} + y^{\frac{2}{3}})$.

31. $(x^{-\frac{1}{2}} + y^{-\frac{1}{2}})(x^{-\frac{1}{2}} - y^{-\frac{1}{2}})$.

32. $(x^{\frac{1}{3}}y^{\frac{1}{4}}z^{\frac{1}{5}} : x^{-\frac{1}{3}}y^{-\frac{1}{4}}z^{-\frac{1}{5}})^{30}$.

33. $\left(m^{-2} - \dfrac{1}{n^2}\right) : \left(\dfrac{1}{m} - n^{-1}\right)$.

34. $(a^{-3} + a^3)(a^3 - a^{-3})$.

35. $(a^2x^2 - y^{-3})(a^2x^2 + y^{-3})$.

36. $(x^{-a}y^{-b}z^{-c} : x^ay^bz^c)^{-abc}$.

37. $(2^{-2}x^{-4} - x^{-2}y^{-3} + y^{-6})^{\frac{1}{2}}$.

38. $[(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}})]^2$.

39. $ax^{\frac{2}{3}} \cdot a^{\frac{1}{3}}x^{\frac{1}{3}} \cdot a^{\frac{1}{6}}x^{\frac{1}{6}} \cdot a^{\frac{5}{6}} : x^{\frac{1}{6}}$.

40. $(x^{\frac{m}{n}} + y^{\frac{n}{m}})^2 \cdot (x^{\frac{m}{n}} - y^{\frac{n}{m}})^2$.

41. $(x^{-m}-1)^3 \cdot (x^{-m}+1)^3.$

42. $(a-b):(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}).$

43. $(a^{-2}\sqrt{x}-2\sqrt[4]{x}:a+1)^{\frac{1}{2}}.$

44. $\sqrt[3]{8\,a^{\frac{3}{7}}+12\,a^{\frac{2}{7}}+6\,a^{\frac{1}{7}}+1}.$

45. $(a^{\frac{1}{2}}b^{\frac{2}{3}}+a^{\frac{2}{3}}b^{\frac{1}{2}})(a^{\frac{1}{2}}b^{\frac{2}{3}}-a^{\frac{2}{3}}b^{\frac{1}{2}}).$

46. $(a^{\frac{7}{8}}b^{\frac{5}{6}}-a^{\frac{3}{4}}b^{\frac{5}{7}}+4\,a^{\frac{3}{8}}b^{\frac{2}{5}}):a^{\frac{2}{3}}b^{\frac{4}{5}}.$

47. $2\,a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}\cdot 3\,a^{\frac{1}{3}}b^{\frac{1}{4}}c^{\frac{1}{5}}\cdot 4\,a^{\frac{1}{6}}b^{\frac{5}{12}}c^{\frac{11}{20}}.$

48. $(a^{-2}b^{-4}+c^{-6}+2\,a^{-1}b^{-2}\div c^3)^{\frac{1}{2}}.$

49. $(a^{-10}-a^{-5}+1):(a^{-2}-a^{-1}+1).$

50. $(4\,x^{-4}+11\,x^{-2}-45):(2\,x^{-1}-3).$

51. $(x^{-1}+y^{-1}+z^{-1})(x^{-1}+y^{-1}-z^{-1}).$

52. $(3^{-2}x^{-4}-2^{-4}y^{-2})(3^{-2}x^{-4}+2^{-4}y^{-2}).$

53. $(a^{\frac{1}{2}}+b^{\frac{1}{2}}+a^{-\frac{1}{2}}b)(ab^{-\frac{1}{2}}-a^{\frac{1}{2}}+b^{\frac{1}{2}}).$

54. $(a^{-4}x^{-4}-b^{-2}y^{-2}):(a^{-2}x^{-2}+b^{-1}y^{-1}).$

55. $(4\,x^{\frac{4}{3}}y^{\frac{5}{3}}-9\,x^{\frac{8}{5}}y^{\frac{7}{4}}):(2\,x^{\frac{2}{3}}y^{\frac{5}{6}}+3\,x^{\frac{4}{5}}y^{\frac{7}{8}}).$

56. $(x^{-3}+2\,x^{-2}y^{-1}-3\,y^{-3}):(x^{-1}-y^{-1}).$

57. $(x^{-2}-3\,x^{-\frac{4}{3}}\sqrt[3]{y^{-2}}+3\,x^{-\frac{2}{3}}y^{-\frac{4}{3}}-y^{-2})^{\frac{1}{3}}.$

58. $\sqrt[3]{8^{-1}x^3-3\cdot 2^{-1}x^2y^{\frac{1}{2}}+6\,xy-8\,y\sqrt{y}}.$

59. $3\,a^{-\frac{3}{4}}:5\,a^{-\frac{5}{6}},\ x^2:\{[x^{-\frac{1}{2}}y^{-\frac{1}{3}}(x^2y^2)^{\frac{2}{3}}]^{-\frac{1}{2}}\}^{-6}.$

60. $(x^2+2\,xy+y^2)(x^{-2}-2\,x^{-1}y^{-1}+y^{-2}).$

61. $(16\,x^{-3}+6\,x^{-2}+5\,x^{-1}-6):(2\,x^{-1}-1).$

62. $(a^{\frac{2}{3}}-a^{\frac{1}{3}}+1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}})(a^{\frac{1}{3}}+1+a^{-\frac{1}{3}}).$

63. $\sqrt[3]{4\,x^{-1}y^2z^{\frac{1}{2}}}:[1:\sqrt{12\,x^3y^{-\frac{2}{3}}z^2}\cdot\sqrt[12]{108\,x^{-3}y^2z^{-4}}].$

64. $(x^{-3}+3\,x^{-2}y+3\,x^{-1}y^2+y^3)(x^{-2}+2\,x^{-1}y+y^2).$

65. $(x^{-5}-2x^{-4}-4x^{-3}+19x^{-2}-31x^{-1}+15):(x^{-3}-7\,x^{-1}+5).$

VI. IRRATIONAL NUMBERS. SURDS

220. Rational and irrational algebraic expressions have already been defined (§ 78). But in algebra it is often necessary to use numbers which are irrational.

221. A **rational number** is a number expressible as the quotient of two integers.

E.g., $3 = \frac{3}{1}$, $0.666 \cdots = \frac{2}{3}$, $\frac{5}{7}$.

222. An **irrational number** is a number which is not rational.

E.g., $2^{\frac{1}{2}}$ or $\sqrt{2}$, $(1 + 2^{\frac{1}{2}})^{\frac{1}{3}}$ or $\sqrt[3]{1+\sqrt{2}}$, $\sqrt{-1}$.

223. Irrational numbers which are not even roots of negative numbers are often called *surds*.

224. Surds are classified as follows:

1. According to the root index, as

quadratic, or of the second order,				as	$\sqrt{5}$,
cubic,	"	"	third	" "	$\sqrt[3]{7}$,
quartic, or biquadratic,				"	$\sqrt[4]{x}$,
quintic,				"	$\sqrt[5]{5}$,
sextic,				"	$\sqrt[6]{a}$, etc.

2. **Similar or dissimilar** (if they have a single term), according as the surd factors are or are not the same.

E.g., $2\sqrt{3}$, $4\sqrt{3}$, $-7\sqrt{3}$ are similar surds.
$2\sqrt{3}$, $3\sqrt{2}$ are dissimilar surds.

3. **Pure or mixed** (if they have a single term), according as they do not or do contain either rational factors or dissimilar surd factors.

E. g., $\sqrt{3}$ is a pure surd, but $2\sqrt{3}$ and $\sqrt{5} \cdot \sqrt[3]{3}$ are mixed surds.

4. According to the number of terms in the expression when simplified, as

monomial surds, as $\sqrt{2}$, $3\sqrt[3]{2}$,
binomial " " $\sqrt{2}+\sqrt[3]{5}$, $5+\sqrt{2}$,
trinomial " " $2+\sqrt{3}+\sqrt[4]{7}$,

and, in general, **polynomial** surds.

5. According to simplicity. A surd is said to be in its *simplest form* when the index is as small as possible, and when the expression under the radical sign is integral and contains as a factor no power of the same degree as the index.

E.g., $\sqrt{9}$, $\sqrt[4]{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{a^3x}$, are not in the simplest form. For

$$\sqrt{9}=3,$$
$$\sqrt[4]{4}=\sqrt{\sqrt{4}}=\sqrt{2},$$
$$\sqrt{\tfrac{1}{2}}=\sqrt{\tfrac{2}{4}}=\sqrt{\tfrac{1}{4}\cdot 2}=\tfrac{1}{2}\sqrt{2}.$$
$$\sqrt[3]{a^3x}=a\sqrt[3]{x}.$$

The fractional exponent is, in general, more convenient in all operations involving surds. The two forms of the radical symbol are used here in order that both may be familiar.

225. Convention as to signs. When we consider an expression like $\sqrt{4}+\sqrt{9}$ we see that it reduces to $(\pm 2)+(\pm 3)$, and hence to

$$+2+3=5,$$
$$+2-3=-1,$$
$$-2+3=1,$$
$$-2-3=-5.$$

But for simplicity it is agreed among mathematicians that in expressions of this kind *only the absolute values of the roots shall be considered*, unless the contrary is stated.

Hence, $\sqrt{4}+\sqrt{9}=2+3=5.$

228. Since in multiplying surds it is desirable to have them of the same order, it is often necessary *to reduce several surds to equivalent surds of the same order,* the order always being as low as possible.

E.g., $\sqrt{2}\cdot\sqrt[3]{3} = 2^{\frac{1}{2}}\cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}}\cdot 3^{\frac{2}{6}} = (2^3\cdot 3^2)^{\frac{1}{6}} = \sqrt[6]{8\cdot 9} = \sqrt[6]{72}.$

ILLUSTRATIVE PROBLEMS

1. Reduce $3\sqrt[3]{5}$ to a pure surd.

$$3\sqrt[3]{5} = \sqrt[3]{3^3\cdot 5} = \sqrt[3]{135}. \qquad \S\ 226$$

2. Reduce $\sqrt{\frac{12}{13}}$ to its simplest form.

$$\sqrt{\frac{12}{13}} = \sqrt{\frac{12\cdot 13}{13^2}} = \sqrt{\frac{4\cdot 3\cdot 13}{13^2}} = \frac{2}{13}\sqrt{39}. \qquad \S\ 227$$

3. Reduce $\sqrt[4]{6}$ and $\sqrt[3]{2}$ to equivalent surds of the same order.

$6^{\frac{1}{4}}$ and $2^{\frac{1}{3}}$ can evidently be reduced to the order indicated by $4\cdot 3$.

$$6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}. \qquad \S\ 207$$

$$2^{\frac{1}{3}} = 2^{\frac{4}{12}} = \sqrt[12]{2^4} = \sqrt[12]{16}.$$

4. Reduce $\sqrt{a^2x^2 + b^2x^2}$ to its simplest form.

$$\sqrt{a^2x^2 + b^2x^2} = \sqrt{x^2(a^2 + b^2)} = x\sqrt{a^2 + b^2}.$$

EXERCISE XCII

Reduce the numbers in Exs. 1–10 to the *form* of surds of the orders indicated.

1.	5,	3d order.	2.	2,	6th order.
3.	$\frac{1}{3}$,	4th "	4.	10,	5th "
5.	11,	2d "	6.	12,	3d "
7.	-2,	2d "	8.	-5,	3d "
9.	3,	5th "	10.	-2,	6th "

Reduce the expressions in Exs. 11–28 to pure surds.

11. $2\sqrt{3}$. **12.** $ab^{\frac{1}{2}}c$. **13.** $2\sqrt[5]{2}$.

14. $5 \cdot 2^{\frac{1}{3}}$. **15.** $3\sqrt{2}$. **16.** $4\sqrt{a}$.

17. $ab\sqrt{cd}$. **18.** $a^{\frac{1}{2}}b^{\frac{1}{2}}c$. **19.** $a\sqrt{2a^3}$.

20. $3\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5}$. **21.** $a^2b\sqrt[3]{ab}$. **22.** $x\sqrt{x+y}$.

23. $(x+y)\sqrt{x-y}$. **24.** $(x^2+y^2)\sqrt{x+y}$.

25. $(x-y)\sqrt{\dfrac{x-y}{x+y}}$. **26.** $(a-b)\sqrt{\dfrac{1}{a^2-b^2}}$.

27. $(a^2+b^2)\sqrt{\dfrac{1}{a^2+b^2}}$. **28.** $(x^2-2)\sqrt{\dfrac{1}{x^2-2}}$.

Reduce the expressions in Exs. 29–38 to the *form* of surds of the orders indicated.

29. $\sqrt[3]{abc^2}$, 9th order. **30.** $\sqrt[7]{a^6}$, 14th order.

31. $\sqrt[3]{5}$, 30th " **32.** $3^{\frac{1}{3}}$, 15th "

33. $5^{\frac{1}{4}}$, 20th " **34.** $10^{\frac{1}{3}}$, 15th "

35. $\sqrt[4]{4}$, 8th " **36.** $\sqrt[15]{5}$, 60th "

37. $\sqrt{2}$, 4th " **38.** $\sqrt[3]{2}$, 6th "

Reduce the expressions in Exs. 39–46 to equivalent surds of the same order, the order being as low as possible in each case:

39. $\sqrt{a}$, $\sqrt[3]{b}$. **40.** $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[7]{2}$.

41. $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{4}}$. **42.** $\sqrt[3]{4}$, $\sqrt[4]{3}$, $\sqrt[12]{5}$.

43. $a^{\frac{1}{2}}b^{\frac{2}{3}}$, $a^{\frac{2}{3}}b^{\frac{1}{2}}$. **44.** $\sqrt[5]{2}$, $\sqrt[2]{5}$, $\sqrt[20]{3}$.

45. $7^{\frac{1}{2}}$, $9^{\frac{1}{6}}$, $11^{\frac{1}{9}}$. **46.** 2, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$.

Reduce the following expressions to their simplest forms:

47. $\sqrt{25}$. **48.** $\sqrt{32}$. **49.** $\sqrt{75}$. **50.** $\sqrt{162}$.

51. $\sqrt{180}$. **52.** $\sqrt{243}$. **53.** $\sqrt{175}$. **54.** $\sqrt{144}$.

55. $\sqrt{a^2b}$. **56.** $\sqrt{a^2b^3}$. **57.** $\sqrt{4\,ab^3}$. **58.** $\sqrt{81\,x^5y^7}$.

59. $2\sqrt{4\,a^2b^3c^4}$. **60.** $3\sqrt{9\,x^3y^4z^2}$. **61.** $5\sqrt{2\,x^7y^{10}z}$.

62. $4\sqrt{81\,x^{10}y^{12}z^{14}}$. **63.** $\sqrt{3\,a^3+a^2}$. **64.** $\sqrt{x^4y+x^3y^2}$.

65. $\sqrt[3]{8\,a^2b^3c^4}$. **66.** $\sqrt[3]{40\,x^2y^3z^9}$.

67. $\sqrt[3]{54\,a^4b^4c^4}$. **68.** $\sqrt[3]{27\,a^5b^9c^{12}}$.

69. $\sqrt[3]{x^4+ax^5}$. **70.** $(x^7+x^6y)^{\frac{1}{3}}$.

71. $\sqrt[3]{m^6x+m^7}$. **72.** $\sqrt[3]{(a+b)^6y^9}$.

73. $a\sqrt[3]{a^4+a^6x}$. **74.** $\sqrt[3]{a^5x^5+a^7x^7}$.

75. $(x^4+x^5y+x^6y^2)^{\frac{1}{2}}$. **76.** $[(a+b)(a^2-b^2)]^{\frac{1}{2}}$.

77. $\sqrt{a^3-2\,a^2b+ab^2}$. **78.** $\sqrt{ab^2c^3(a+b+c)^3}$.

79. $\sqrt[3]{(a^2+ab)(a^4+2\,a^3b+a^2b^2)}$. **80.** $\sqrt[3]{a^4+3\,a^3b+3\,a^2b^2+ab^3}$.

81. $\sqrt{\frac{1}{3}}$. **82.** $\sqrt{\frac{2}{5}}$. **83.** $\sqrt[3]{\frac{1}{2}}$.

84. $\sqrt[3]{\frac{2}{a}}$. **85.** $\sqrt[4]{\frac{1}{8}}$. **86.** $\sqrt{\frac{a^3}{b}}$.

87. $\sqrt[5]{\frac{1}{16}}$. **88.** $\sqrt{\frac{2\,x}{9\,y}}$. **89.** $\sqrt[3]{\frac{a^2b}{ab^2}}$.

90. $\sqrt{\frac{2\,a}{7\,b}}$. **91.** $\sqrt{\frac{abc}{def}}$. **92.** $\sqrt[3]{\frac{4\,a}{9\,b^2}}$.

93. $\sqrt[3]{\frac{abc^2}{xyz^2}}$. **94.** $\sqrt[5]{\frac{a}{81\,b^3}}$. **95.** $\sqrt[3]{\frac{1}{3\,ab^2}}$.

96. $\sqrt[4]{\frac{3\,a^5x^3}{4\,y^3z}}$. **97.** $\sqrt[4]{\frac{2\,ab^2}{27\,a^3b}}$. **98.** $\sqrt[5]{\frac{x^2y^3z}{27\,a^3b^4}}$.

229. Addition and subtraction of surds. Irrational expressions may evidently be added and subtracted the same as rational expressions, by taking advantage of some convenient unit.

ILLUSTRATIVE PROBLEMS

1. Required the sum of $\sqrt{24}$, $\sqrt{54}$, and $-\sqrt{96}$. Here we have, each surd being reduced to its simplest form,

$$\begin{aligned} \sqrt{24} &= \sqrt{4\cdot 6} &&= 2\sqrt{6} \\ \sqrt{54} &= \sqrt{9\cdot 6} &&= 3\sqrt{6} \\ -\sqrt{96} &= -\sqrt{16\cdot 6} &&= \underline{-4\sqrt{6}} \end{aligned}$$

Hence, the sum is $\sqrt{6}$

2. Required the sum of $\sqrt{8}$, $\sqrt{27}$, $-2\sqrt{2}$, and $\sqrt{48}$. Here we have $2\sqrt{2}+3\sqrt{3}-2\sqrt{2}+4\sqrt{3}=7\sqrt{3}$.

3. Add the following:

$$\begin{array}{rrc} & & \textit{Check.} \\ a\sqrt{x}+ & b\sqrt[3]{x}-c\sqrt[n]{z} & 1 \\ - & c\sqrt[3]{x}+c\sqrt[n]{z} & 0 \\ a\sqrt{x}+ & b\sqrt[3]{x}+c\sqrt[n]{z} & \underline{3} \\ \hline 2a\sqrt{x}+ & (2b-c)\sqrt[3]{x}+c\sqrt[n]{z} & 4 \end{array}$$

In general, however, the sums of surds can only be indicated as $\sqrt{3}+\sqrt[3]{7}$, $-\sqrt[n]{a}+\sqrt[m]{b}$.

EXERCISE XCIII

Find the sum of the expressions in Exs. 1–12.

1. $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{3}{16}}$.
2. $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{9}}$, $\sqrt{\frac{1}{8}}$.
3. $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$.
4. $\sqrt{24}$, $\sqrt{54}$, $\sqrt{6}$.
5. $\sqrt{\frac{1}{5}}$, $\sqrt{\frac{4}{5}}$, $\frac{1}{5}\sqrt{245}$.
6. $\sqrt[3]{5}$, $\sqrt[3]{40}$, $\sqrt[3]{1080}$.
7. $\sqrt{3}$, $\sqrt{75}$, $\sqrt{108}$.
8. $\sqrt{5}$, $\sqrt{125}$, $\sqrt{500}$.
9. $\sqrt{63}$, $\sqrt{28}$, $\sqrt{175}$.
10. $\sqrt{60}$, $\sqrt{135}$, $\sqrt{240}$.
11. $\sqrt[3]{56}$, $\sqrt[3]{189}$, $\sqrt[3]{875}$.
12. $\sqrt[4]{32}$, $\sqrt[4]{162}$, $\sqrt[4]{512}$.

282. When both equations are quadratic. In this case, x can be found in terms of y in either equation, but, *in general,* the value will involve y^2. Then the value of x substituted in the other equation will involve y^4, and hence *the result will be an equation of the fourth degree.*

E.g., given the system $\qquad x^2 - y^2 = -3.$

$$2x^2 + 3x + y = 7.$$

From the first equation

$$x = \pm\sqrt{y^2 - 3}.$$

Substituting in the second,

$$2(y^2 - 3) \pm 3\sqrt{y^2 - 3} + y = 7.$$

Isolating the radical, squaring, and reducing, we have

$$2y^4 + 2y^3 - 30y^2 - 13y + 98 = 0,$$

an equation of the fourth degree.

283. Hence, *in general, two simultaneous quadratic equations involving two unknown quantities cannot be solved by means of quadratics.*

It is only in special cases that such systems admit of solution by quadratics, and four pairs of roots should always be expected.

A few of the more common of these special cases will now be considered.

284. When one equation is homogeneous. In this case a solution is always possible. For if $ax^2 + bxy + cy^2 = 0$ is the homogeneous equation we can divide by y^2 and have $a \cdot \frac{x^2}{y^2} + b \cdot \frac{x}{y} + c = 0$, a quadratic in $\frac{x}{y}$. Hence, $\frac{x}{y}$ can be found and x will then be known as a multiple of y, and this value can then be substituted in the other equation.

ILLUSTRATIVE PROBLEMS

1. Solve the system

$$1. \quad x^2 - \tfrac{5}{2}xy + y^2 = 0.$$
$$2. \quad x^2 + 3x - 4y + 4 = 0.$$

3. From (1) $$\left(\frac{x}{y}\right)^2 - \frac{5}{2}\left(\frac{x}{y}\right) + 1 = 0.$$

4. $$\therefore 2\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 2 = 0,$$

or $$\left(2 \cdot \frac{x}{y} - 1\right)\left(\frac{x}{y} - 2\right) = 0.$$

5. $$\therefore \frac{x}{y} = \frac{1}{2}, \text{ or } 2, \text{ and } x = \frac{y}{2}, \text{ or } 2y.$$

Substituting $x = \dfrac{y}{2}$ in equation 2, we have

6. $$\frac{y^2}{4} + \frac{3y}{2} - 4y + 4 = 0.$$

7. $$\therefore y^2 - 10y + 16 = 0.$$

8. $$\therefore y = 2, \text{ or } 8, \text{ and } \therefore x = \frac{y}{2} = 1, \text{ or } 4.$$

Substituting $x = 2y$ in equation 2 and reducing, we have

9. $$y^2 + \tfrac{1}{2}y + 1 = 0.$$

10. $$\therefore y = -\tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{-15}.$$

11. $$\therefore x = 2y = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-15}.$$

12. $$\therefore x = 1,\ 4,\ -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-15},\ -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-15},$$

and $$y = 2,\ 8,\ -\tfrac{1}{4} + \tfrac{1}{4}\sqrt{-15},\ -\tfrac{1}{4} - \tfrac{1}{4}\sqrt{-15},$$

these roots being taken in pairs in the order indicated.

2. Solve the system

$$1. \quad x^2 - xy = 0.$$
$$2. \quad x^2 + 3xy + 5y^2 = 36.$$

From (1) $x = y$, or 0.

Substituting y for x, $9y^2 = 36$ and $y = \pm 2$,

whence $x = \pm 2$.

Substituting 0 for x, $5y^2 = 36$,

$$\therefore y = \pm \tfrac{6}{5}\sqrt{5}.$$

EXERCISE CXVIII

Solve the following systems of equations:

1. $x^2 + y^2 - bxy = 0.$
$x + y = a.$

2. $3x^2 + 3xy - y^2 = 5.$
$x^2 - 2xy + y^2 = 0.$

3. $5x^2 + 4xy - y^2 = 0.$
$x^2 + x + y = 5.$

4. $x^2 + xy + x - y = -2.$
$2x^2 - xy - y^2 = 0.$

5. $x^2 + 3xy + 3x - y = 2.$
$x^2 + 2xy - 3y^2 = 0.$

6. $x^2 - y^2 + x + y = \frac{18}{36}.$
$36(x^2 + y^2) = 97xy.$

7. $2x^2 + 3xy + 4y = 18.$
$x^2 + 4xy = 12y^2.$

8. $3x^2 + 4xy + 3x - y = 3.$
$x^2 + xy = 0.$

9. $x^2 + 4x + 3y + y^2 = -2.$
$x(x + 2y) - 15y^2 = 0.$

10. $x(x + y) + y(y + x) = 4xy.$
$x(x + y) + y + x = 24.$

11. $x^2 - 3x + 4y + 2xy = 24.$
$x^2 + 3xy = 4y^2.$

12. $147x^2 + 196xy + 57y^2 = 0.$
$x^2 + 2xy + 33 = 0.$

285. When both equations are homogeneous except for the absolute terms. In this case a solution is always possible by quadratics. For if

$$a_1x^2 + b_1xy + c_1y^2 = d_1,$$

and
$$a_2x^2 + b_2xy + c_2y^2 = d_2,$$

we can multiply both members of the first by d_2, and of the second by d_1, and subtract, and

$$(a_1d_2 - a_2d_1)x^2 + (b_1d_2 - b_2d_1)xy + (c_1d_2 - c_2d_1)y^2 = 0.$$

This may now be treated as in § 284.

ILLUSTRATIVE PROBLEM

Solve the system

$$1.\quad x^2 + 3\,xy - 2\,y^2 = 2.$$
$$2.\quad 2\,x^2 - 5\,xy + 6\,y^2 = 3.$$

Multiplying both members of equation 1 by 3, and of equation 2 by 2, and subtracting, we have:

3. $$x^2 - 19\,xy + 18\,y^2 = 0.$$

This equation is easily solved by factoring. If it were not, we should divide by y^2 and proceed as in § 284.

4. $$\therefore\ (x - 18\,y)(x - y) = 0.$$

5. $$\therefore\ x = 18\,y, \text{ or } y.$$

Substituting $18\,y$ for x in (1), we have

6. $$324\,y^2 + 54\,y^2 - 2\,y^2 = 2.$$

7. $$\therefore\ y = \pm\tfrac{1}{2}\sqrt{\tfrac{1}{47}} = \pm\tfrac{1}{94}\sqrt{47},$$

and $$x = 18\,y \qquad = \pm\tfrac{9}{47}\sqrt{47}.$$

Substituting y for x in (1), we have

8. $$y^2 + 3\,y^2 - 2\,y = 2.$$

9. $$\therefore\ y = \pm 1, \text{ whence } x = \pm 1.$$

Check for $x = \pm\frac{9}{47}\sqrt{47}$, $y = \pm\frac{1}{94}\sqrt{47}$.

$$\tfrac{81}{47} + \tfrac{27}{94} - \tfrac{2}{188} = 2.$$
$$\tfrac{162}{47} - \tfrac{45}{94} + \tfrac{3}{94} = 3.$$

286. Since §§ 284 and 285 depend upon finding the value of $\frac{x}{y}$, or of $\frac{y}{x}$, we can also solve by letting $\frac{y}{x} = v$, or $y = vx$. then finding v.

E.g., in the preceding example we had the system

1. $$x^2 + 3xy - 2y^2 = 2.$$

2. $$2x^2 - 5xy + 6y^2 = 3.$$

Let $\frac{y}{x} = v$, or $y = vx$. Then, from (1), we have

3. $$x^2 + 3vx^2 - 2v^2x^2 = 2.$$

4. $$\therefore\ x^2 = \frac{2}{1 + 3v - 2v^2}.$$

Similarly, from (2), we have

5. $$2x^2 - 5vx^2 + 6v^2x^2 = 3.$$

6. $$\therefore\ x^2 = \frac{3}{2 - 5v + 6v^2}.$$

Equating the values of x^2,

7. $$\frac{2}{1 + 3v - 2v^2} = \frac{3}{2 - 5v + 6v^2}.$$

Reducing,

8. $$18v^2 - 19v + 1 = 0,$$

or $$(18v - 1)(v - 1) = 0.$$

9. $$\therefore\ v = \tfrac{1}{18}, \text{ or } 1.$$

10. $$\therefore\ y = vx = \tfrac{1}{18}x, \text{ or } x.$$

This is substantially the same as step 5 of the preceding solution (p. 299), and the rest of the work is as given there.

In the same way we may let $\frac{x}{y} = u$, or $x = uy$. We should then have, from equation (1),

$$u^2y^2 + 3uy^2 - 2y^2 = 2.$$

$$\therefore\ y^2 = \frac{2}{u^2 + 3u - 2}.$$

Similarly, from (2) $$y^2 = \frac{3}{2u^2 - 5u + 6}.$$

Equating these values of y^2, u can be found as above.

EXERCISE CXIX

Solve the following systems of equations:

1. $x^2 + 2xy = 39.$
   $xy + 2y^2 = 65.$

2. $x^2 + 3xy = 2.$
   $3y^2 + xy = 1.$

3. $x^2 + 3xy = 54.$
   $xy + 4y^2 = 115.$

4. $2x^2 + 3xy = 27.$
   $xy + y^2 = 4.$

5. $m^2x^2 + n^2y^2 = q^2.$
   $\frac{x^2}{a^2} = \frac{y^2}{b^2}.$

6. $7x^2 - 5xy = 18.$
   $\frac{x^2}{y^2} + 3 = \frac{7}{y^2}.$

7. $3xy + y^2 - 18 = 0.$
   $4x^2 + xy - 7 = 0.$

8. $x^2 - xy + y^2 = 21.$
   $y^2 - 2xy = -15.$

9. $x^2 + xy + y^2 = 139.$
   $5y^2 - 4xy = -75.$

10. $ax^2 + b(x^2 + y^2) = m.$
    $cy^2 + d(x^2 + y^2) = n.$

11. $x^2 - xy + 2y^2 = 0.$
    $3y^2 - 2(5y - 4) = 0.$

12. $3x^2 - 5xy + 2y^2 = 14.$
    $2x^2 - 5xy + 3y^2 = 6.$

13. $2x^2 + 2xy + y^2 = 73.$
    $x^2 + xy + 2y^2 = 74.$

14. $32y^2 - 2xy - 11 = 0.$
    $x^2 + 4y^2 = 10.$

15. $3x^2 + 13xy + 8y^2 = 162.$
    $x^2 - xy + y^2 = 7.$

16. $(3x + y)(3y + x) = 384.$
    $(x - y)(x + y) = 40.$

17. $3x^2 + 4xy + 5y^2 - 48 = 0.$
    $4x^2 + 5xy - 36 = 0.$

18. $2x^2 + 3xy - 3y^2 + 124 = 0.$
    $7x^2 - xy - y^2 + 49 = 0.$

287. When the equations are symmetric with respect to the two unknown quantities, that is, when the unknown quantities can be interchanged without affecting the equations.

Such equations are

$$x + y = 3,$$
$$xy = 2.$$

In these cases a solution is always possible by quadratics. The solution is accomplished by letting $x = u + v$, and $y = u - v$, and first solving for u and v.

ILLUSTRATIVE PROBLEMS

1. Solve the system

$$1.\quad x^2 + 3xy + y^2 = 41.$$
$$2.\quad x^2 + y^2 + x + y = 32.$$

Let $x = u + v$ and $y = u - v$. Then, by substituting in (1), we have

3. $$5u^2 - v^2 = 41, \text{ or } v^2 = 5u^2 - 41.$$

Substituting in (2),

4. $$u^2 + v^2 + u = 16.$$

Substituting here the value of v^2 from (3),

5. $$6u^2 + u - 57 = 0,$$

or $$(6u + 19)(u - 3) = 0.$$

6. $$\therefore u = -\tfrac{19}{6}, \text{ or } 3.$$

Substituting this value of u in (3),

7. $$v = \pm \tfrac{1}{6}\sqrt{329}, \text{ or } \pm 2.$$

8. $\therefore x = u + v = \dfrac{-19 \pm \sqrt{329}}{6}$, 5, or 1, four values, as we should expect (§ 282).

9. Since the x and y may be interchanged without affecting the given equations, y must have the same values, always arranged so that $x + y$ shall equal $2u$.

10. $\therefore$ for $$x = \frac{-19 + \sqrt{329}}{6},\ \frac{-19 - \sqrt{329}}{6},\ 5,\ 1,$$

we have $$y = \frac{-19 - \sqrt{329}}{6},\ \frac{-19 + \sqrt{329}}{6},\ 1,\ 5.$$

2. Solve the system

$$1.\quad x^2 + y^2 = 25.$$
$$2.\quad x - y = 1.$$

This system is symmetric with respect to x and $-y$. Hence y must have the same absolute values as x, but with opposite signs.

3. Let $x = u + v$ and $y = u - v$. Then from (1),

$$2u^2 + 2v^2 = 25.$$

4. From (2) $\quad 2v = 1.$

$$\therefore v = \tfrac{1}{2}.$$

5. From (3) and (4), $\quad 2u^2 + \tfrac{1}{2} = 25,$

$$\therefore u^2 = \tfrac{49}{4},$$
$$\therefore u = \pm \tfrac{7}{2}.$$

6. $$\therefore x = u + v = \pm \tfrac{7}{2} + \tfrac{1}{2} = 4, \text{ or } -3.$$

7. $$\therefore y = -x = -4, \text{ or } 3.$$

These values must evidently be taken in pairs such that $x - y = 1$; *i.e.*, when $x = 4$, y must equal 3; when $x = -3$, y must equal -4.

EXERCISE CXX

Solve the following systems of equations:

1. $x^2 + y^2 = 41.$
   $x - y = 1.$

2. $x^2 - xy + y^2 = 3.$
   $x^2 + xy + y^2 = 7.$

3. $x(x + y) - 40 = 0.$
   $y(y + x) - 60 = 0.$

4. $x + \sqrt{xy} + y = 14.$
   $x^2 + xy + y^2 = 84.$

5. $x^2 + xy + y^2 = 19.$
   $x + y = 5.$

6. $x^2 - xy + y^2 - 49 = 0.$
   $x + y - 13 = 0.$

7. $x^2 + y^2 + 3(x + y) = 4.$
   $3x^2 + 4xy + 3y^2 = 3.$

8. $2x^2 + xy + 2y^2 = 79.58.$
   $x^2 - 2xy + y^2 = 21.29.$

9. $\dfrac{1}{x+2} + \dfrac{1}{y+2} = \dfrac{55}{63}.$
   $\dfrac{1}{x} + \dfrac{1}{y} = 7.$

10. $x - y = 4.$
    $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{26}{5}.$

288. When equations above the second degree are involved. *In general, such systems cannot be solved by quadratics,* although they can be solved in special cases.

E.g., $$x^3 + x^2y + y^3 = 11.$$
$$x - y = -1.$$

Here $x = y - 1$; hence,

$$(y-1)^3 + (y-1)^2y + y^3 = 11,$$

or $$3y^3 - 5y^2 + 4y - 12 = 0, \text{ a cubic equation.}$$

Now a cubic equation may sometimes be solved by factoring, as here, for this reduces (§ 89) to

$$(y-2)(3y^2 + y + 6) = 0,$$

whence $$y = 2, \text{ or } \tfrac{1}{6}(-1 \pm \sqrt{-71}),$$

whence $$x = 1, \text{ `` } \tfrac{1}{6}(-7 \pm \sqrt{-71}).$$

These results check, although the labor of substituting the complex number is too great to make it worth while, except in such cases as the teacher may direct.

289. If the equations are symmetric with respect to the unknown quantities, they often yield to the method given in § 287.

E.g., to solve the system

1. $$x^3 + y^3 = 91.$$

2. $$x + y = 7.$$

Let $x = u + v$, $y = u - v$. Then

3. $$2u^3 + 6uv^2 = 91, \text{ from (1).}$$

4. $$u = \tfrac{7}{2}, \text{ `` (2).}$$

5. $$\therefore \tfrac{343}{4} + 21v^2 = 91, \text{ and } v = \pm\tfrac{1}{2}.$$

6. $\therefore x = u + v = 4$, or 3, and $\therefore y = 3$, or 4, by symmetry.

This system is easily solved in other ways, as by dividing the members of (1) by the members of (2), etc.

EXERCISE CXXI

Solve the following systems of equations:

1. $x^3+y^3=9.$ $x+y=3.$

2. $x^4+y^4=97.$ $x+y=1.$

3. $x^3+y^3=72.$ $x+y=6.$

4. $x^3+y^3=341.$ $x+y=11.$

5. $x^3+y^3=218.$ $x+y=2.$

6. $x^3+y^3=98.$ $x-y=-8.$

7. $x^4+y^4=337.$ $x+y=7.$

8. $x^4+y^4=641.$ $x-y=7.$

9. $x^4+y^4=337.$ $x-y=1.$

10. $x^3-y^3=279.$ $x-y=3.$

11. $x^5+y^5=4149.$ $x+y=9.$

12. $x^2+y^2+xy(x+y)=154.$ $x^3+y^3-3(x^2+y^2)=50.$

13. $\frac{1}{x^3}-\frac{1}{y^3}=19.$ $\frac{1}{x}-\frac{1}{y}=1.$

14. $\frac{1}{x^3}+\frac{1}{y^3}=35.$ $\frac{1}{x}+\frac{1}{y}=5.$

15. $\frac{1}{x^3}+\frac{1}{y^3}=\frac{35}{216}.$ $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}.$

16. $\frac{1}{x^3}+\frac{1}{y^3}=133.$ $\frac{1}{x}+\frac{1}{y}=8.$

17. $\frac{1}{x^4}+\frac{1}{y^4}=641.$ $\frac{1}{x}+\frac{1}{y}=7.$

18. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}.$ $\frac{1}{x}+\frac{1}{y}=\frac{10}{xy}.$

19. $\sqrt{x+y}+\frac{4}{\sqrt{x+y}}-4=0.$ $\frac{x^2+y^2}{xy}=\frac{34}{15}.$

290. Special devices will frequently suggest themselves, but it is not worth while to attempt to classify them. A few are given in the following illustrative problems.

1. Solve the system

$$1. \quad x^2y^2 + xy - 6 = 0.$$

$$2. \quad x^2 + y^2 = 5.$$

3. From (1), $(xy - 2)(xy + 3) = 0$, whence $xy = 2$, or -3.

4. Adding $2xy = 4$ or -6 to and subtracting it from, the respective members of (2), we have

5. $$x^2 + 2xy + y^2 = 9, \text{ or } -1.$$
$$x^2 - 2xy + y^2 = 1, \text{ " } 11.$$

6. $$\therefore x + y = \pm 3, \text{ or } \pm\sqrt{-1}.$$
$$x - y = \pm 1, \text{ " } \pm\sqrt{11}.$$

Adding, and dividing by 2,

7. $x = \dfrac{\pm 3 \pm 1}{2}$, or $\dfrac{\pm\sqrt{-1} \pm \sqrt{11}}{2}$

$= 2, 1, -1, -2, \dfrac{\sqrt{-1} + \sqrt{11}}{2}, \dfrac{\sqrt{-1} - \sqrt{11}}{2}, \dfrac{-\sqrt{-1} + \sqrt{11}}{2}, \dfrac{-\sqrt{-1} - \sqrt{11}}{2}.$

On account of symmetry, y must have the same values, arranged so as to satisfy step (6).

$\therefore y = 1, 2, -2, -1,$

$\dfrac{\sqrt{-1} - \sqrt{11}}{2}, \dfrac{\sqrt{-1} + \sqrt{11}}{2}, \dfrac{-\sqrt{-1} - \sqrt{11}}{2}, \dfrac{-\sqrt{-1} + \sqrt{11}}{2}.$

Check. For the last results.

Substituting $x = \dfrac{-\sqrt{-1} - \sqrt{11}}{2}$, $y = \dfrac{-\sqrt{-1} + \sqrt{11}}{2}$ in equations (1) and (2),

$$\left(\frac{-\sqrt{-1}-\sqrt{11}}{2} \cdot \frac{-\sqrt{-1}+\sqrt{11}}{2}\right)^2 + \frac{-\sqrt{-1}-\sqrt{11}}{2} \cdot \frac{-\sqrt{-1}+\sqrt{11}}{2} - 6$$
$$= (-3)^2 + (-3) - 6 = 9 - 3 - 6 = 0.$$

$$\left(\frac{-\sqrt{-1}-\sqrt{11}}{2}\right)^2 + \left(\frac{-\sqrt{-1}+\sqrt{11}}{2}\right)^2 = \frac{10+2\sqrt{-11}}{4} + \frac{10-2\sqrt{-11}}{4} = 5.$$

2. Solve the system

$$1.\quad x = a\sqrt{x+y}.$$
$$2.\quad y = b\sqrt{x+y}.$$

Adding,

3. $$x + y = (a+b)\sqrt{x+y}, \text{ or}$$
$$x + y - (a+b)\sqrt{x+y} = 0, \text{ or}$$

4. $$\sqrt{x+y}\,(\sqrt{x+y} - \overline{a+b}) = 0.$$

5. $$\therefore\ \sqrt{x+y} = 0, \text{ or } a+b.$$

Substituting in (1) and (2),

$$x = 0, \text{ or } a(a+b).$$
$$y = 0, \text{ `` } b(a+b).$$

The results check.

3. Solve the system

$$1.\quad x^4 + x^2y^2 + y^4 = 481.$$
$$2.\quad x^2 + xy + y^2 = 37.$$

Factoring (1), by § 86,

3. $$(x^2 + xy + y^2)(x^2 - xy + y^2) = 481.$$

4. $$\therefore\ 37(x^2 - xy + y^2) = 481, \text{ or}$$
$$x^2 - xy + y^2 = 13.$$

Subtracting from (2),

5. $$2xy = 24, \text{ whence } xy = 12.$$

Adding to (2), and subtracting from (4),

6. $$x^2 + 2xy + y^2 = 49.$$
$$x^2 - 2xy + y^2 = 1.$$

7. $$\therefore\ x + y = \pm 7.$$
$$x - y = \pm 1.$$

8. $$\therefore\ x = 4,\ -4,\ 3,\ -3,\quad y = 3,\ -3,\ 4,\ -4.$$

MISCELLANEOUS EXERCISE CXXII

Solve the following systems of equations:

1. $x^3 + y^3 = b.$
   $x + y = a.$

2. $x + y + xy = 34.$
   $x^2 + y^2 - x - y = 42.$

3. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x.$
   $x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$

4. $x^2 - xy + y^2 = 124.$
   $x^2 - y^2 = 44.$

5. $x^2 + y^2 = 25x^2y^2.$
   $12xy = 1.$

6. $x^2 + y^2 - x - y = \frac{7}{4}.$
   $xy = 1.$

7. $\sqrt{x} + \sqrt{y} = 12.$
   $x^2 + y^2 = 3026.$

8. $3(x^2 + y^2) = 10(x + y).$
   $9(x^4 + y^4) = 34(x^3 + y^3).$

9. $x^2 + 4xy + 4y^2 = 6.$
   $3x^2 + 8y^2 = 14.$

10. $x + y = 5.$
    $(x^2 + y^2)(x^3 + y^3) = 455.$

11. $4x^2 + y^2 + 4x + 2y = 6.$
    $2xy = 1.$

12. $(x^2 + xy + y^2)\sqrt{x^2 + y^2} = 185.$
    $(x^2 - xy + y^2)\sqrt{x^2 + y^2} = 65.$

13. $x + y - 2\sqrt{xy} - \sqrt{x} + \sqrt{y} = 2.$
    $\sqrt{x} + \sqrt{y} = 7.$

14. $\sqrt{x} + \sqrt{y} = x - y = x - \sqrt{xy} + y.$

15. $x^2 - 6xy + 9y^2 - 4x + 12y = -4.$
    $x^2 - 2xy + 3y^2 - 4x + 5y = 53.$

16. $\dfrac{x^2y + 12}{x} = y(1 + x).$
    $\dfrac{x^2 + xy + 2y^2}{x + y} = \dfrac{32}{x + y} + y.$

II. THREE OR MORE UNKNOWN QUANTITIES

291. *In general, three simultaneous quadratic equations involving three unknown quantities cannot be solved by quadratics.*

Many special cases, however, admit of such solution.

The same is true if one equation is linear and the other two are quadratic, or if one is of a degree higher than 2.

If, however, two are linear and the other quadratic, a solution is possible by quadratics, as in illustrative problem 2 on p. 310.

ILLUSTRATIVE PROBLEMS

1. Solve the system

$$1.\ 3\,xy = 2\,x + 2\,y.$$
$$2.\ 2\,yz = 3\,y + 2\,z.$$
$$3.\ 4\,zx = 5\,z - 3\,x.$$

Dividing both members of (1), (2), (3), by xy, yz, zx, respectively, we have

4. $$3 = \frac{2}{y} + \frac{2}{x}.$$

5. $$2 = \frac{2}{y} + \frac{3}{z}.$$

6. $$4 = \frac{5}{x} - \frac{3}{z}.$$

Adding (5) and (6),

7. $$6 = \frac{5}{x} + \frac{2}{y}.$$

Eliminating y, from (4) and (7),

8. $$3 = \frac{3}{x},\text{ whence } x = 1.$$

$$\therefore y = 2,\ z = 3.$$

Check.
$$6 = 2 + 4,$$
$$12 = 6 + 6,$$
$$12 = 15 - 3.$$

2. Solve the system

$$1.\quad x + y - 2z = -9.$$
$$2.\quad 3x + 2y + z = 9.$$
$$3.\quad x^2 + y^2 + z^2 = 30.$$

Eliminating z from (1) and (2),

4. $$y = \frac{9 - 7x}{5}.$$

Eliminating y from (1) and (2),

5. $$z = \frac{27 - x}{5}.$$

Substituting (4) and (5) in (3), and reducing,

6. $$5x^2 - 12x + 4 = 0, \text{ or}$$
$$(x - 2)(5x - 2) = 0.$$

7. $$\therefore x = 2, \text{ or } \tfrac{2}{5}.$$
$$\therefore y = -1, \text{ or } \tfrac{31}{25}.$$
$$\therefore z = 5, \text{ or } 5\tfrac{8}{25}.$$

Check for the second set of values.

$$\tfrac{2}{5} + \tfrac{31}{25} - 10\tfrac{16}{25} = -9.$$
$$\tfrac{6}{5} + \tfrac{62}{25} + 5\tfrac{8}{25} = 9.$$
$$\tfrac{4}{25} + \tfrac{961}{625} + \tfrac{17689}{625} = \tfrac{18750}{625} = 30.$$

EXERCISE CXXIII

Solve the following systems of equations:

1. $4y^3 = 9xz.$
   $x^3 = 36yz.$
   $9z^3 = 4xy.$

2. $x^2 + y^2 = 2.$
   $y^2 + z^2 = 2.$
   $z^2 + x^2 = 2.$

3. $2y - yz = 4.$
   $2z - zx = 9.$
   $2x - xy = 16.$

4. $x^2 + y^2 + xy = 19.$
   $y^2 + z^2 + yz = 37.$
   $z^2 + x^2 + zx = 28.$

5. $xy = 15.$
$yz = 40.$
$zx = 24.$

6. $xy = 2.$
$yz = 38.$
$zx = 19.$

7. $x^2 + xy + y^2 = 12.$
$y^2 + yz + z^2 = 12.$
$z(z + y) = xy.$

8. $2x - 3y + 5z = 0.$
$3x + 6y - 7z = 0.$
$x^2 + 2y^2 + 3z^2 = 6.$

9. $(y + 1)(z + 1) = 63.$
$(x + 1)(z + 1) = 45.$
$(x + 1)(y + 1) = 35.$

10. $(y + z)(z + x) = 210.$
$(z + x)(x + y) = 182.$
$(x + y)(y + z) = 195.$

11. $\dfrac{xy}{x + y} = 1.$
$\dfrac{zx}{z + x} = 2.$
$\dfrac{yz}{y + z} = 3.$

12. $\dfrac{xyz}{x + y} = \dfrac{9}{2}.$
$\dfrac{xyz}{y + z} = 2.$
$\dfrac{xyz}{z + x} = \dfrac{18}{7}.$

13. $x + y = \dfrac{1}{z}.$
$y + z = \dfrac{1}{x}.$
$z + x = \dfrac{1}{y}.$

14. $\dfrac{x^2 + y^2}{xyz} = \dfrac{5}{6}.$
$\dfrac{z^2 + x^2}{xyz} = \dfrac{5}{3}.$
$\dfrac{y^2 + z^2}{xyz} = \dfrac{13}{6}.$

15. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{23}{15}.$
$\dfrac{3}{x} + \dfrac{6}{y} = 5.$
$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{xy}.$

16. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 9.$
$\dfrac{2}{x} + \dfrac{2}{y} = 13.$
$\dfrac{3}{x} + \dfrac{8}{y} = \dfrac{5}{xy}.$

17. $(y + z)^2 - x^2 = (z + x)^2 - y^2 = (x + y)^2 - z^2 = 3.$

REVIEW EXERCISE CXXIV

1. Solve the system of equations $xy = 7$, $x^2 + y^2 = 50$.

2. Solve the equation $\sqrt{\frac{x}{4} - \frac{1}{2}} - \frac{1}{2} = \sqrt{\frac{x}{3}} - 1$.

3. Solve the system of equations $x + y = 11$, $x^2 - y^2 = 99$.

4. Solve the system of equations

$$x + y = a + b,\ x/a - y/b = a/b - b/a.$$

5. Multiply $x^{-\frac{2}{3}} - 2(xy)^{-\frac{1}{3}} + y^{-\frac{2}{3}}$ by $\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{y}}$.

6. Solve the system of equations

$$4y + 5z = 11,\ 3z + 6x = 9,\ 8x - 3y = 4.$$

7. Form the equation whose roots are 0, $\sqrt{-1}$, $\sqrt{-1}$.

Solve the following systems of equations :

8. $(x - y)(x^2 - y^2) = 160.$
   $(x + y)(x^2 + y^2) = 580.$

9. $x + y = 2xy = x^2 - y^2.$

10. $x + y + (x + y)^{\frac{1}{2}} - 12 = 0.$
    $x^2 + y^2 - 45 = 0.$

11. $\dfrac{1}{x + y} + \dfrac{1}{x - y} = \dfrac{3}{4}.$
    $2x^3 + 6xy^2 = \dfrac{9}{64}(x^2 - y^2)^3.$

12. $(3x + 4y)(7x - 2y) + 3x + 4y = 44.$
    $(3x + 4y)(7x - 2y) - 7x + 2y = 30.$

13. $17(x + y)^{-\frac{1}{2}} - 7(x + y)^{\frac{1}{2}}x^{-1} = 10x(x + y)^{-\frac{3}{2}}.$
    $(x - y)^{\frac{1}{2}} = y - 1.$

III. PROBLEMS INVOLVING QUADRATICS

EXERCISE CXXV

1. The difference of two numbers is 11, the sum of their squares 901. What are the numbers?

2. The sum of two numbers is 30, the sum of their squares 458. What are the numbers?

3. Find two numbers whose sum, whose product, and the difference of whose squares are all equal.

4. The sum of the squares of two numbers is 421, the difference of the squares 29. What are the numbers?

5. A certain fraction equals 0.625, and the product of the numerator and denominator is 14,440. Required the fraction.

6. The sum of the areas of two circles is 24,640 sq. in., and the sum of their radii is 112 in. Required the lengths of their radii.

7. The product of the numbers $2x3$ and $4y6$, in which x and y stand for the tens' digit, x being twice y, is 103,518. What are the tens' digits?

8. If a certain two-figure number, the sum of whose digits is 11, is multiplied by the units' digit, the product is 296. Required the number.

9. Three successive integers are so related that the square of the greatest equals the sum of the squares of the other two. Required the numbers.

10. Separate the number 102 into three parts such that the product of the first and third shall be 102 times the second, and the third shall be $\frac{3}{2}$ of the first.

11. Two cubes have together the volume 407 cu. in., and the sum of one edge of the one and one of the other is 11 in. Required the volume of each.

12. If the product of two numbers is increased by their sum, the result is 89; if the product is diminished by their sum, the result is 51. Required the numbers.

13. One of the sides forming the right angle of a right-angled triangle is $\frac{3}{7}$ the other, and the area of the triangle is 5082 sq. in. Required the lengths of the sides.

14. There are two numbers such that the product of the first and 1 more than the second is 660, and the product of the second and 1 less than the first is 609. What are the numbers?

15. A sum of money at interest for 5 yrs. amounts to \$4600. Had the rate been increased 1% it would have amounted to \$40 more than this in 4 yrs. Required the capital and the rate.

16. The product of the numbers $x17$ and $2y2$, in which x stands for the hundreds' digit of the first and y for the tens' of the second, and in which $y = x + 3$, is 83,054. Required the values of x and y.

17. Find a two-figure number such that the product of the two digits is half the number, and such that the difference between the number and the number with the digits interchanged is $\frac{3}{2}$ of the product of the two digits.

18. In going 1732.5 yds. the front wheel of a wagon makes 165 revolutions more than the rear wheel; but if the circumference of each wheel were 27 in. more, the front wheel would, in going the same distance, make only 112 revolutions more than the rear one. Required the circumference of each wheel.

19. The floor of a certain room has 210 sq. ft., each of the two side walls 135 sq. ft., and each of the two end walls 126 sq. ft. Required the dimensions of the room.

20. A certain cloth loses $\frac{1}{8}$ in length and $\frac{1}{16}$ in width by shrinking. Required the length and width of a piece which loses 3.68 sq. yds., and which has its perimeter decreased 3.4 yds. by shrinking.

21. A rectangular field is 119 yds. long and 19 yds. wide. How much must the width be decreased and the length increased in order that the area shall remain the same while the perimeter is increased 24 yds.?

22. Two points move, each at a uniform rate, on the arms of a right angle toward the vertex, from two points 50 in. and 60 in., respectively, from the vertex. After 7 secs. the points are $2\sqrt{377}$ in. apart, and after 9 secs. they are $2\sqrt{193}$ in. apart. Required the rate of each.

23. There are two lines such that if they are made the sides of a right-angled triangle the hypotenuse is 17 in.; but if one be made the hypotenuse and the other a side, the remaining side is such that the square constructed upon it contains 161 sq. in. How long are the two lines?

24. There is a fraction whose numerator being increased by 2 and denominator diminished by 2, the result is the reciprocal of the fraction; but if the denominator is increased by 2 and the numerator diminished by 2, the result is $1\frac{1}{15}$ less than the reciprocal. Required the fraction.

25. If the numerator of a fraction is decreased by 2, and the new fraction added to the original one, the sum is $1\frac{2}{3}$; if the denominator is decreased by 2, and the new fraction added to the original one, the sum is $2\frac{1}{60}$. Required the fraction.

CHAPTER XVII

INEQUALITIES

292. Having given two real and unequal numbers, a and b, $a - b$ cannot be zero. If $a - b$ is positive, a is said to be *greater than* b; if negative, a is said to be *less than* b.

E.g., $3 > 2$ because $3 - 2$ is positive,

$-2 > -3$ " $-2 - (-3)$ is positive,

$-3 < -2$ " $-3 - (-2)$ is negative

If $a > 0$, then a is positive, and if

$a < 0$, " " " negative.

293. The inequalities $a > b$, $c > d$ are called **inequalities in the same sense,** and similarly for $a < b$, $c < d$. But $a > b$, $c < d$ are called **inequalities in the opposite sense,** and similarly for $a < b$, $c > d$.

294. In this chapter the letters used to represent numbers will be understood to represent *positive and real finite numbers,* except as the minus sign indicates a negative number.

295. Just as we distinguish two classes of equalities, (1) equations and (2) identities, so in inequalities we have two classes, (1) those which are true only for particular values of a quantity called the unknown quantity, and (2) those which are true for all values of the letters.

E.g., $x + 2 > 3$ is true only when $x > 1$, but $a + b > b$ is always true.

296. The axioms of inequalities. The following axioms have already been assumed and used:

Ax. 4. *If equals are added to unequals, the sums are unequal in the same sense.*

That is, if $a = b$

and $c > d$,

then $a + c > b + d$.

Ax. 5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

That is, if $c > d$

and $a = b$,

then $c - a > d - b$.

Three important theorems of inequalities will now be proved, the first two corresponding to Axs. 6 and 8.

297. Theorem. *If unequals are multiplied by equals, the products are unequal in the same or in the opposite sense, according as the multiplier is positive or negative.*

That is, if $c > d$,

then $2c > 2d$,

and $-2c < -2d$.

Proof. 1. If $a > b$, then $a - b$ is positive. § 292

2. Then $k(a - b)$ is positive,
and $-k(a - b)$ is negative. § 292

3. $\therefore ka - kb$ is positive,
and $-ka - (-kb)$ is negative.

4. $\therefore ka > kb$,
and $-ka < -kb$. § 292

298. Theorem. *If* $a > b$, *then* $a^m > b^m$, m *being a positive integer.*

Proof. 1. $a - b$ is positive.

2. $\therefore (a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1})(a - b)$ is positive, because the multiplier is evidently a positive quantity.

3. $\therefore$ $a^m - b^m$ is positive, because this is the product of the expressions.

4. $$\therefore a^m > b^m. \qquad \text{§ 292}$$

299. Theorem. *If* $a \neq b$, $a^2 + b^2 > 2ab$.

Proof. 1. $(a - b)^2 > 0$, because $(a - b)^2$ is positive, being the square of a real number. It is not 0, for $a \neq b$.

2. $$\therefore a^2 - 2ab + b^2 > 0.$$

3. $$\therefore a^2 + b^2 > 2ab.$$

Evidently $a^2 + b^2 = 2ab$, if $a = b$.

This theorem is the most important of inequalities. The following is one of the interesting results derived from it:

The sum of any positive number and its reciprocal is, in general, greater than 2.

For if $n =$ any positive number,

then $\frac{1}{n} =$ its reciprocal.

Then $$n + \frac{1}{n} = \frac{n^2 + 1}{n}.$$

But $$\frac{n^2 + 1}{n} > \frac{2n}{n} = 2.$$

The single exception is the case of $n = 1$, when $n + \frac{1}{n} = 2$.

ILLUSTRATIVE PROBLEMS

1. Prove that $x^2 > 2x - 1$, if $x \neq 1$.

We have $x^2 + 1 > 2x$, by § 299.

2. Prove that $x^{p+q} + y^{p+q} > x^p y^q + x^q y^p$, if $x \neq y$.

1. This is true if $x^{p+q} - x^p y^q + y^{p+q} - x^q y^p$ is positive.

2. Or if $x^p (x^q - y^q) - y^p (x^q - y^q)$ is positive.

3. Or if $(x^p - y^p)(x^q - y^q)$ is positive.

4. But both factors are positive if $x > y$, and both factors are negative if $x < y$, and in either case their product is positive.

3. Which is greater, $2 + \sqrt{3}$, or $2.5 + \sqrt{2}$.

1. $$2 + \sqrt{3} \gtreqless 2.5 + \sqrt{2}, \text{ according as}$$

2. $$7 + 4\sqrt{3} \gtreqless 8\tfrac{1}{4} + 5\sqrt{2}, \text{ squaring.} \qquad \text{§ 298}$$

3. Or as $$-1\tfrac{1}{4} + 4\sqrt{3} \gtreqless 5\sqrt{2}. \qquad \text{Ax. 5}$$

4. Or as $$49\tfrac{9}{16} - 10\sqrt{3} \gtreqless 50. \qquad \text{§ 298}$$

5. Or as $$-10\sqrt{3} \gtreqless \tfrac{7}{16}. \qquad \text{Ax. 5}$$

6. But a negative number is less than a positive one.

$$\therefore 2 + \sqrt{3} < 2.5 + \sqrt{2}.$$

4. Solve the inequality $2x - \frac{x}{3} + \frac{1}{2} > 3x - \frac{1}{3} + \frac{x}{6}$.

1. $$12x - 2x + 3 > 18x - 2 + x. \qquad \text{§ 297}$$

2. $$\therefore -9x > -5. \qquad \text{Ax. 5}$$

3. $$\therefore x < {}^5.$$

Check. **If $x = \frac{5}{9}$, the inequality becomes an equation. If $x > \frac{5}{9}$, the sense of the inequality is reversed.**

5. Solve the inequality $x^2 - 5x + 6 < 0$.

1. $(x - 2)(x - 3) < 0$, and hence is negative.

2. The smaller factor, $x - 3$, is negative, and the other positive.

3. $\therefore x > 2$ and $x < 3$, or $2 < x < 3$.

6. Find values of x and y that will satisfy the following system:

$$1.\quad x+2y>6.$$

$$2.\quad x+y=5.$$

3. **Eliminating x,**

$$y>1.$$

4. **Substituting in (2), x must be less than 4, because $y>1$.**

$\therefore$ **any solution of (2), in which $x<4$ and $y>1$ satisfies the system.**

***E.g.*, $x=3$, $y=2$, or $x=3\frac{1}{2}$, $y=1\frac{1}{2}$.**

EXERCISE CXXVI

Prove that the inequalities stated in Exs. 1–6 are true, in general.

1. $(x+y)^2>4xy$.
2. $x^3+1>x^2+x$.
3. $2x^2+13>12x$.
4. $x^2-y^2>2(x+y-1)$.
5. $a^2+2b^2+c^2>2b(a+c)$.
6. $(a+b)(b+c)(c+a)>8abc$.

Solve the inequalities in Exs. 7–14.

7. $x^2-3x<10$.
8. $x^2+x>12$.
9. $x^2+3x>-2$.
10. $x^2+5x>-6$.
11. $x(x-10)<11$.
12. $x^2-5x>-4$.
13. $\dfrac{x-3}{x-4}>0$.
14. $5x+2>3x+\dfrac{x}{2}-7$.

Find values of x and y that will satisfy the following systems:

15. $3x+y=31$.
    $x-y>8$.
16. $x+by=13$.
    $7x-y>4$.
17. $2x+3y=23$.
    $x+y>8$.
18. $x+5y=52$.
    $2x-y<6$.

CHAPTER XVIII

RATIO, VARIATION, PROPORTION

I. RATIO

300. The **ratio** of one number, a, to another number, b, of the same kind, is the quotient $\frac{a}{b}$.

Thus, the ratio of \$2 to \$5 is $\frac{\$2}{\$5}$, or $\frac{2}{5}$, or 0.4, but there is no ratio of \$2 to 5 ft., or \$10 to 2. Here, as elsewhere in algebra, however, the letters are understood to represent pure (abstract) numbers.

A ratio may be expressed by any symbol of division, *e.g.*, by the fractional form, by $\div$, by /, or by :; but the symbols generally used are the fraction and the colon, as $\frac{a}{b}$, or $a : b$.

301. In the ratio $a : b$, a is called the **antecedent** and b the **consequent**.

302. The ratio $b : a$ is called the **inverse** of the ratio $a : b$.

303. If two variable quantities, x, y, have a constant ratio, r, one is said to **vary** as the other. *E.g.*, a circumference varies as the diameter.

If $\frac{x}{y} = r$, then $x = ry$. The expression "x varies as y" is sometimes written $x \propto y$, meaning that $x = ry$.

If $x = r \cdot \frac{1}{y}$, x is said to vary inversely as y.

ILLUSTRATIVE PROBLEM

If the ratio of x^2 to 3 is 27, find the value of x.

$\because \frac{x^2}{3} = 27$, $\therefore x^2 = 3 \cdot 27 = 81$, $\therefore x = \pm 9$, and each value checks.

EXERCISE CXXVII

1. The ratio of 625 to x^3 is 5. Find x.

Find the value of x from the ratios in Exs. 2–7.

2. $4 : x^2 = 9$.
3. $x^2 : 27 = 300$.
4. $x = \frac{1}{144} : x$.
5. $\frac{x^2}{63} = 7$.
6. $\frac{36}{x} = x$.
7. $\frac{x}{2} = \frac{8}{x}$.

Find the value of x from the ratios in Exs. 8–12.

8. $\frac{x^2}{15} = 2.4$.
9. $\frac{3}{x^3} = \frac{81}{8}$.
10. $\frac{49x^4}{432} = \frac{3}{49}$.
11. $7 : x = 4.9$.
12. $x^3 : 5 = \frac{25}{8}$.

13. One cube is 1.2 times as high as another. Find the ratio of (1) their surfaces, (2) their volumes.

304. Applications in business. Of the numerous applications of ratio in business, only a few can be mentioned, and not all of these commonly make use of the word "ratio."

In computing interest, the simple interest varies as the time, if the rate is constant; as the rate, if the time is constant; as the product of the rate and the number representing the time in years (if the rate is by the year), if neither is constant.

I.e., for twice the rate, the interest is twice as much, if the time is constant; for twice the time, the interest is twice as much, if the rate is constant; but for twice the time and 1.5 times the rate, the interest is $2 \cdot 1.5$ times as much.

The common expressions "2 out of 3," "2 to 5," "6 per cent" (merely 6 out of 100) are only other methods of stating the following ratios of a part to a whole, $\frac{2}{3}$, $\frac{2}{7}$, $\frac{6}{100}$, or the following ratios of the two parts, $\frac{2}{1}$, $\frac{2}{5}$, $\frac{6}{94}$.

E.g., to divide $100 between A and B so that A shall receive $2 out of every $3, is to divide it into two parts

(1) having the ratio 2 : 1, or

(2) so that A's share shall have to the whole the ratio 2 : 3, or

(3) so that B's share shall have to the whole the ratio 1 : 3.

EXERCISE CXXVIII

1. Divide $1000 so that A shall have $7 out of every $8.

2. Divide $625 so that A shall have $1 out of every $5.

3. Divide $546 so that A shall have $1 out of every $6.

4. Divide $500 between A and B so that A shall have $0.25 as often as B has $1.25.

5. The area of the United States is 3,501,000 sq. mi., and the area of Russia is 8,644,100 sq. mi. Express the ratio of the former to the latter, correct to 0.01.

6. The white population of the United States in 1780 was 2,383,000; in 1790, 3,177,257; in 1880, 43,402,970; in 1890, 54,983,890. What is the ratio of the population in 1790 to that in 1780? in 1890 to that in 1880?

7. The depths of three artesian wells are as follows: A 220 m., B 395 m., C 543 m.; the temperatures of the water from these depths are: A 19.75° C., B 25.33° C., C 30.50° C. From these observations, is it correct to say that the increase of temperature is proportional to the increase of depth? If not, what should be the temperature at C to have this law hold?

305. A ratio is called a **ratio of greater inequality**, of **equality**, or of **less inequality**, according as the antecedent is greater than, equal to, or less than the consequent.

306. Theorem. *A ratio of greater inequality is diminished, a ratio of equality is unchanged in value, and a ratio of less inequality is increased by adding any positive quantity to both terms.*

Given the ratio $a : b$, and p any positive quantity.

To prove that $\frac{a+p}{b+p} \underset{>}{\overset{<}{=}} \frac{a}{b}$ according as $a \underset{<}{\overset{>}{=}} b$.

Proof. 1. $\frac{a+p}{b+p} \underset{>}{\overset{<}{=}} \frac{a}{b}$ according as

$$ab + pb \underset{>}{\overset{<}{=}} ab + ap. \qquad \text{§ 297, Ax. 6}$$

2. Or, as $pb \underset{>}{\overset{<}{=}} ap$, or as $b \underset{>}{\overset{<}{=}} a$. § 296

3. *I.e.*, as $a \underset{<}{\overset{>}{=}} b$.

307. Theorem. *If* $\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{d}} = \frac{\mathrm{e}}{\mathrm{f}} = \cdots$, *then each of these ratios equals* $\frac{\mathrm{a} + \mathrm{c} + \mathrm{e} + \cdots}{\mathrm{b} + \mathrm{d} + \mathrm{f} + \cdots}$.

Proof. 1. Let $\frac{a}{b} = k$. Then $k = \frac{c}{d} = \frac{e}{f} = \cdots$.

2. $\therefore a = kb,$

$c = kd,$

$e = kf, \cdots.$

3. $\therefore a + c + e + \cdots = k(b + d + f + \cdots)$. **Ax. 2**

4. $\therefore \frac{a + c + e + \cdots}{b + d + f + \cdots} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$. **Ax. 7**

EXERCISE CXXIX

1. Prove that the product of two ratios of greater inequality is greater than either.

2. Consider Ex. 1 for two ratios of equality; of less inequality. Then state the general theorem and prove it.

3. Find the value of x, knowing that if x is subtracted from both terms of the ratio $\frac{1}{2}$ the ratio is squared.

4. Is the value of a ratio changed by raising both terms to the same power? State the general theorem and prove it.

5. Prove (or show that it has been proved) that the value of a ratio is not changed by multiplying both terms by the same number.

6. As in § 306, consider the effect of subtracting from both terms of a ratio any positive number not greater than the less term. State the theorem and prove it.

7. Which is the greater ratio, $\dfrac{a+5b}{a+6b}$ or $\dfrac{a+6b}{a+7b}$?

8. Which is the greater ratio, $\dfrac{x-2y}{y-2x}$ or $\dfrac{x-3y}{3y-2x}$?

9. Which is the greater ratio, $\dfrac{a+b+c}{a-b-c}$ or $\dfrac{a-b+c}{a+b-c}$?

10. If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, prove that $\dfrac{a^2+b^2+c^2}{ab+bc+cd}=\dfrac{ab+bc+cd}{b^2+c^2+d^2}$.

11. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$, prove that $k=\dfrac{3a+5c-6e}{3b+5d-6f}$.

12. If $\dfrac{a}{b}>\dfrac{c}{d}$, the letters standing for positive numbers, prove that $\dfrac{a}{b}>\sqrt{\dfrac{a^2+c^2}{b^2+d^2}}>\dfrac{c}{d}$.

II. VARIATION

308. Theorem. *If* $x \propto y$ *and* $y \propto z$, *then* $x \propto z$.

Proof. 1. If $x \propto y$, then $x = ky$, § 303

2. If $y \propto z$, then $y = k'z$. § 303

3. $\therefore x = ky = kk'z$. Substn.

4. $\therefore x \propto z$. § 303

Note that in step 2 we cannot use the same constant as in step 1.

E.g., if the edge of a cube varies as the diagonal of a face, and the diagonal of a face varies as the diagonal of the cube, then the edge must vary as the diagonal of the cube.

309. Theorem. *If* $x \propto yz$, *then* $y \propto \frac{x}{z}$.

Proof. 1. $x = kyz$.

2. $\therefore y = \frac{1}{k} \cdot \frac{x}{z}$. Ax. 7

3. $\therefore y \propto \frac{x}{z}$. § 303

E.g., if the area of a rectangle varies as the product of the (numbers representing the) base and altitude, then the base varies as the quotient of the (number representing the) area divided by the (number representing the) altitude.

310. Theorem. *If* $w \propto x$ *and* $y \propto z$, *then* $wy \propto xz$.

Proof. 1. $w = kx$ and $y = k'z$.

2. $\therefore wy = kk'xz$.

3. $\therefore wy \propto xz$.

E.g., if the surface of a sphere varies as the square of the diameter, and $\frac{1}{3}$ of the radius varies as the radius, then the product of the surface and $\frac{1}{3}$ of the radius varies as the product of the radius and the square of the diameter.

311. Theorem. *If* $x \propto y$ *when* z *is constant, and if* $x \propto z$ *when* y *is constant, then* $x \propto yz$ *when both* y *and* z *vary.*

To understand this statement, consider a simple illustration: The area of a triangle varies as the altitude when the base is constant, and as the base when the altitude is constant; but it varies as the product of their numerical values when both base and altitude vary.

Proof. 1. Let the variations of y and z take place separately.

2. Let x change to x' when y changes to y', z remaining unchanged. Then

$$\because x \propto y, \therefore \frac{x}{x'} = \frac{y}{y'}.$$

3. Let x' change to x'' when y' remains unchanged and z changes to z'. Then

$$\because x \propto z, \therefore \frac{x'}{x''} = \frac{z}{z'}.$$

4. $\therefore \frac{x}{x'} \cdot \frac{x'}{x''}$, or $\frac{x}{x''}$, equals $\frac{y}{y'} \cdot \frac{z}{z'}$, or $\frac{yz}{y'z'}$.

5. *I.e.*, x changes to x'' as yz changes to $y'z'$, or $x \propto yz$.

ILLUSTRATIVE PROBLEMS

1. If $x \propto y$, and if $x = 2$ when $y = 5$, find x when $y = 11$.

$\because x \propto y$ means that $x = ky$, $\therefore 2 = k \cdot 5$, and $k = \frac{2}{5}$. $\therefore x = \frac{2}{5}y$. When $y = 11$, $x = \frac{2}{5} \cdot 11 = 4.4$.

2. The volumes of spheres vary as the cubes of their radii. Two spheres of metal are melted into a single sphere. Required its radius.

1. $v = kr^3$ and $v' = kr'^3$. § 303

2. $\therefore$ the volume of the single sphere is $k(r^3 + r'^3)$.

3. Call v'' this volume, and r'' the radius; then

$$v'' = k(r^3 + r'^3) = kr''^3.$$

4. $\therefore r''^3 = r^3 + r'^3$, and $\therefore r'' = (r^3 + r'^3)^{\frac{1}{3}}$.

EXERCISE CXXX

1. If $x \propto z$ and $y \propto z$, prove that $xy \propto z^2$.

2. If $x \propto z$ and $y \propto z$, prove that $x + y \propto z$.

3. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

4. If $w \propto x$ and $y \propto z$, prove that $w / y \propto x / z$.

5. If $10x + 3y = 7x - 4y$, show that $x \propto y$.

6. If $a^x \propto b^y$, and if $x = 3$ when $y = 5$, prove that $a^{x-3} = b^{y-5}$.

7. If $x \propto y$, and if $x = a$ when $y = b$, find the value of x when $y = c$.

8. If $x \propto y$, and if $x = 7$ when $y = 11$, find the value of x when $y = 7$.

9. If $x \propto y$, prove that $px \propto py$, p being either a constant or a variable.

10. What is the radius of the circle which is equal to the sum of two circles whose radii are 3 and 4, respectively?

11. Prove that the volume of the sphere whose radius is 6 is equal to the sum of ths volumes of three spheres whose radii are 3, 4, and 5, respectively.

12. The illumination from a given source of light varies inversely as the square of the distance. How much farther from an electric light 20 ft. away must a sheet of paper be removed in order to receive half as much light?

13. Kepler showed that the squares of the numbers representing the times of revolution of the planets about the sun vary as the cubes of the numbers representing their distances from the sun. Mars being 1.52369 as far as the earth from the sun, and the time of revolution of the earth being 365.256 das., find the time of revolution of Mars.

III. PROPORTION

312. The equality of two ratios forms a **proportion.**

Thus, $\frac{3}{8} = \frac{4}{6}$, $a : b = c : d$, $x/y = m/n$, are examples of proportion. The symbol :: was formerly much used for $=$.

313. There may be an equality of several ratios, as $1 : 2 = 4 : 8 = 9 : 18$, the term **continued proportion** being applied to such an expression.

Three quantities, a, b, c, are said to be in continued proportion when $a : b = b : c$.

314. There may also be an equality between the products of ratios, as $\frac{3}{8} \cdot \frac{5}{7} = \frac{1}{7} \cdot \frac{10}{8}$, such an expression being called a **compound proportion.**

315. In the proportion $a : b = c : d$, a, b, c, d, are called the **terms**, a and d being called the **extremes** and b and c the **means.** The term d is called the **fourth proportional** to a, b, c.

316. In the proportion $a : b = b : c$, b is called the **mean proportional** between a and c, and c is called the **third proportional** to a and b.

317. If one quantity varies directly as another, the two are said to be **directly proportional,** or simply **proportional.**

E.g., at retail the cost of a given quality of sugar varies directly as the weight; the cost is then proportional to the weight. Thus, at 4 cts. a pound 12 lbs. cost 48 cts., and 4 cts. : 48 cts. = 1 lb. : 12 lbs.

318. If one quantity varies inversely as another, the two are said to be **inversely proportional.**

Thus if $x = \frac{a}{y}$, x and y are said to be inversely proportional.

ILLUSTRATIVE PROBLEMS

1. What are the mean proportionals between 5 and 125?

1. $$\frac{5}{x}=\frac{x}{125}.$$

2. $$\therefore 625 = x^2.$$

3. $$\therefore \pm 25 = x, \text{ and both results check.}$$

2. What is the fourth proportional to 1, 5, 9?

1. $$\frac{1}{5}=\frac{9}{x}.$$

2. $$\therefore x = 5 \cdot 9 = 45, \text{ and the result checks.}$$

3. What number must be added to the numbers 1, 6, 7, 18 so that the sums shall form a proportion?

1. $$\frac{1+x}{6+x}=\frac{7+x}{18+x}.$$

2. $$\therefore 18 + 19x + x^2 = 42 + 13x + x^2.$$

3. $$\therefore x = 4.$$

EXERCISE CXXXI

Find the value of x in Exs. 1–10.

1. $3 : x = 4 : 5.$
2. $x : 11 = 17 : 121.$
3. $4 : 7 = x : 84.$
4. $-x : 13 = 4 : 52.$
5. $x : 5 = 45 : x.$
6. $15 : -1 = 45 : x.$
7. $5 : 9 = x : 108.$
8. $-7 : 9 = -3 : x.$
9. $27 : x = x : 48.$
10. $1.43 : x = 4.01 : 2.$
11. What is the third proportional to $1+\sqrt{-1}$ and -2?
12. What is the fourth proportional to -4, -3, -16?
13. What are the mean proportionals between 1 and 1?
14. What number must be subtracted from 36, 31, 61, 51 so that the remainders shall form a proportion?
15. What number must be subtracted from 50, 45, 75, 65 so that the remainders shall form a proportion?

319. Theorem. *In any proportion in which the numbers are all abstract, the product of the means equals the product of the extremes.*

For if $\frac{a}{b}=\frac{c}{d}$, then, by multiplying by bd,

$$ad = bc. \qquad \text{Ax. 6}$$

320. Theorem. *If the product of two abstract numbers equals the product of two others, either two may be made the means and the other two the extremes of a proportion.*

For if $ad = bc$, then by dividing by bd,

$$\frac{a}{b}=\frac{c}{d}. \qquad \text{Ax. 7}$$

Similarly, $\frac{b}{a}=\frac{d}{c}$, etc.

321. Theorem. *If* $a : b = c : d$, *then* $a : c = b : d$.

The proof is left for the student.

The old mathematical term for the interchange of the means is "alternation." The first proportion is "taken by alternation" to get the second. The term, while of little value, is still used.

322. Theorem. *If* $a : b = c : d$, *then* $b : a = d : c$.

The proof is left for the student.

The old mathematical term for this change is "inversion."

323. Theorem. *If* $a : b = c : d$, *then* $a + b : b = c + d : d$.

The proof is left for the student.

The old mathematical term for this change is "composition."

324. Theorem. *If* $a : b = c : d$, *then* $a - b : b = c - d : d$.

The proof is left for the student.

The old mathematical term for this change is "division."

325. Theorem. *If*

$$a : b = c : d, \text{ then } a + b : a - b = c + d : c - d.$$

Proof. 1. $$\frac{a+b}{b} = \frac{c+d}{d}. \qquad \S\ 323$$

2. $$\frac{a-b}{b} = \frac{c-d}{d}. \qquad \S\ 324$$

3. $$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}. \qquad \text{Ax. } 7$$

4. $$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}. \qquad \S\ 135$$

The old mathematical term for this change is "composition and division."

There is sometimes an advantage in applying this principle in solving fractional equations. *E.g.*, given the equation

$$\frac{x^2 + 3x - 1}{x^2 - 3x + 1} = \frac{x^2 - 4x + 2}{x^2 + 4x - 2}.$$

$$\frac{2x^2}{6x - 2} = \frac{2x^2}{-8x + 4}.$$

$$\therefore x = 0,$$

or $$6x - 2 = -8x + 4,$$

and $$\therefore x = \tfrac{3}{7}.$$

326. Theorem. *The mean proportionals between two numbers are the two square roots of their product.*

Proof. 1. $$\frac{a}{x} = \frac{x}{b}.$$

2. $$\therefore x^2 = ab. \qquad \S\ 319, \text{ or Ax. } 6$$

3. $$\therefore x = \pm\sqrt{ab}. \qquad \text{Ax. } 9$$

ILLUSTRATIVE PROBLEMS

1. If $a : b = c : d$, prove that $a + b + c + d : b + d = c + d : d$.

1. This is true if

$$ad + bd + cd + d^2 = bc + bd + cd + d^2. \qquad \S\ 320$$

2. Or if $\qquad ad = bc. \qquad$ Ax. 3

3. But $\qquad ad = bc. \qquad$ § 319

4. $\therefore$ reverse the process, deriving step 1 from step 3, and the original proportion from step 1.

2. Solve the equation $\dfrac{\sqrt{x+2}+\sqrt{x-3}}{\sqrt{x+2}-\sqrt{x-3}} = 1\frac{1}{2}$.

We may clear of fractions at once, isolate the two radicals, and square; but in this and similar cases § 325 can be used to advantage.

Writing the second member $\frac{3}{2}$ and applying § 325, we have

1. $$\frac{2\sqrt{x+2}}{2\sqrt{x-3}} = \frac{5}{1}.$$

2. $$\therefore \frac{x+2}{x-3} = 25.$$

3. $$\therefore x + 2 = 25x - 75.$$

4. $$\therefore x = \tfrac{77}{24}.$$

Check. Substitute $\frac{77}{24}$ for x in the original equation, and reduce; then

$$\frac{3\sqrt{\frac{5}{6}}}{2\sqrt{\frac{5}{6}}} = 1\tfrac{1}{2}.$$

3. Find a mean proportional between $1 + \sqrt{-1}$ and $-2 - 14\sqrt{-1}$.

1. By § 326 this equals $\pm\sqrt{(1+\sqrt{-1})(-2-14\sqrt{-1})}$

2. $$= \pm\sqrt{12 - 16\sqrt{-1}}$$

3. $$= \pm 2\sqrt{3 - 4\sqrt{-1}}$$

4. $$= \pm 2\sqrt{4 - 2\sqrt{-4} - 1}$$

5. $$= \pm 2(2 - \sqrt{-1}). \qquad \S\ 288$$

[illegible]

1. Find the value of x in [illegible] $- \sqrt{-1} = x :$ [illegible]

2. Find the mean proportional to $1 - \sqrt{2}$ and $1 - 3\sqrt{2}$.

If $a : b = c : d$, prove the statements in Exs. 3–6.

3. [illegible]

4. [illegible]

5. [illegible]

6. [illegible] $: cd - bc$.

7. If $a : b = b : c$, prove that $a + c > 2b$.

8. What are the mean proportionals between $1 + \sqrt{-1}$ and $2(1 - \sqrt{-1})$?

9. If $a : b = b : c$, prove that $(a + c)b$ is a mean proportional between $a^2 + b^2$ and $b^2 + c^2$.

Find the two mean proportionals between the numbers given in Exs. 10–13.

10. 2 and 98.

11. 50 and -2.

12. 3 and 432.

13. -7 and -847.

14. Given $16 - 6x : 3 = 2 + x : x$, to find x.

15. Given $\sqrt{x+7} + \sqrt{x-7} : \sqrt{x+7} - \sqrt{x-7} = 6 : 1$, to find x.

16. Given $1 + x : 13 - x = x - 2 : x^2 - 21 = x + 4 : 37 - x^2$, to find x.

17. Given $a - b : \dfrac{(a+b)^2}{2ab} - 1 = x : a + b + \dfrac{2b^2}{a-b}$, to find x.

18. Given $3a^2 + 2ab - 8b^2 : 5a^2 + 4ab - 12b^2 = x : 5a - 6b$, to find x.

19. Given $\sqrt{x-5} : \sqrt{7+x} = 1 : 2$, to find x.

20. Find the value of x in

$$3 + 4x - x^2 : 3 - 4x + x^2 = 2 + x : 2 - x.$$

21. Given $x : y = a + b - \frac{ab}{a+b} : a - b + \frac{ab}{a-b}$, and $x + y : a^3 = 2 : 1$, to find x and y.

22. If $\frac{ax+cy}{by+dz} = \frac{ay+cz}{bz+dx} = \frac{az+cx}{bx+dy}$, prove that each of these ratios equals $\frac{a+c}{b+d}$.

23. If $\frac{a-b}{ay+bx} = \frac{b-c}{bz+cx} = \frac{c-a}{cy+az} = \frac{a+b+c}{ax+by+cz}$, prove that each of these ratios equals $\frac{1}{x+y+z}$.

CHAPTER XIX

SERIES

327. A **series** is a succession of terms formed according to some common law.

E.g., in the following, each term is formed from the preceding as indicated:

$1, 3, 5, 7, \cdots$, by adding 2;

$7, 3, -1, -5, \cdots$, by subtracting 4, or by adding -4;

$3, 9, 27, 81, \cdots$, by multiplying by 3, or by dividing by $\frac{1}{3}$;

$2, 2, 2, 2, \cdots$, by adding 0, or by multiplying by 1.

In the series $0, 1, 1, 2, 3, 5, 8, 13, \cdots$, each term after the first two is found by adding the two preceding terms.

328. An **arithmetic series** (also called an arithmetic progression) is a series in which each term after the first is found by adding a constant to the preceding term.

E.g., $-7, -1, 5, 11, \cdots$, the constant being 6,

$2, 2, 2, 2, \cdots$, " " " 0,

$98, 66, 34, 2, \cdots$, " " " -32.

329. A **geometric series** (also called a geometric progression) is a series in which each term after the first is formed by multiplying the preceding term by a constant.

E.g., $3, -6, +12, -24, \cdots$, the constant being -2,

$10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$, " " " $\frac{1}{2}$,

$2, 2, 2, 2, \cdots$, " " " 1.

330. The terms between the first and last are called the **means** of the series.

I. ARITHMETIC SERIES

331. Symbols. The following are in common use:

n, the number of terms of the series.

s, " sum " " " " "

a, the 1st term, and l, the nth or last term.

d, the constant which added to any term gives the next; d is usually called the *difference*.

332. Formulas. There are two formulas in arithmetic series of such importance as to be considered fundamental.

I. $$l = a + (n-1)d.$$

The second term $= a + d$, by definition.

$\therefore$ the third term $= a + 2d$.

$\therefore$ the fourth term $= a + 3d$.

$\vdots$ $\vdots$

$\therefore$ the nth term $= a + (n-1)d$.

3. Or $l = a + (n-1)d$.

E.g., the 50th term in the series 2, 7, 12, 17, $\cdots$ is $2 + 49 \cdot 5 = 247$.

II. $$s = \frac{n(a+l)}{2}.$$

Proof. 1. $s = a + (a+d) + (a+2d) + \cdots (l-d) + l$.

2. Hence, $s = l + (l-d) + (l-2d) + \cdots (a+d) + a$, by reversing the order.

3. $\therefore 2s = (a+l) + (a+l) + \cdots (a+l)$. **Ax. 2**

4. $\therefore 2s = n(a+l)$, $\because$ there is an $(a+l)$ in step 3 for each of the n terms in step 1.

5. $$\therefore s = \frac{n(a+l)}{2}.$$

E.g., the sum of the first 50 terms of the series 2, 7, 12, 17, $\cdots$, of which l has just been found, is

$$\frac{50(2+247)}{2} = 6225.$$

333. It is evident that from formulas I and II various others can be deduced.

E.g., given d, l, s, to find n. The problem merely reduces to that of eliminating a from I and II, and solving for n.

1. From I, $a = l - (n-1)d.$
2. Substituting in II, $s = \dfrac{n[2l - (n-1)d]}{2}.$
3. $\therefore\ n^2 - \dfrac{2l+d}{d} \cdot n + \dfrac{2s}{d} = 0.$
4. $\therefore\ n = \dfrac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$ § 264

ILLUSTRATIVE PROBLEMS

1. Which term of the series 25, 22, 19, ⋯ is -125?

1. Given $a = 25$, $d = -3$, $l = -125$, to find n.
2. $\because\ l = a + (n-1)d,\ -125 = 25 + (n-1)(-3).$
3. Solving, $n = 51$.

2. Insert arithmetic means between 5 and 41 so that the 4th of these means shall have to the next to the last, less 1, the ratio 1 : 2.

1. The means are $5 + d$, $5 + 2d$, $\cdots 41 - 2d$, $41 - d$.
2. $\therefore\ \dfrac{5+4d}{41-2d-1} = \dfrac{1}{2}.$
3. $\therefore\ d = 3$, and the means are 8, 11, 14, 17, ⋯ 35, 38.

3. The sum of three numbers of an arithmetic series is 12 and the sum of their squares is 56. Find the numbers.

In this and similar cases it is advisable to take $x - y$, x, $x + y$, the common difference being y. In the case of four numbers it is advisable to take $x - 3y$, $x - y$, $x + y$, $x + 3y$, $2y$ being the difference.

1. $(x-y) + x + (x+y) = 12,\ \therefore x = 4.$
2. $(x-y)^2 + x^2 + (x+y)^2 = 56,\ \therefore 3x^2 + 2y^2 = 56.$
3. $\therefore\ y = \pm 2.$
4. $\therefore$ the numbers are 4 ∓ 2, 4, 4 ± 2; that is, 2, 4, 6, or 6, 4, 2.

334. The following table gives the various formulas of arithmetic series, and these should be worked out from formulas I and II by the student.

	Given	To find	Result
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{(a-\frac{1}{2}d)^2 + 2ds}.$
3	$a\ n\ s$		$l = 2s/n - a.$
4	$d\ n\ s$		$l = s/n + (n-1)d/2.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{1}{2}(l+a) + (l^2 - a^2)/2d.$
7	$a\ n\ l$		$s = \frac{1}{2}n(a+l).$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = s/n - \frac{1}{2}(n-1)d.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l+\frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = 2s/n - l.$
13	$a\ n\ l$	d	$d = (l-a)/(n-1).$
14	$a\ n\ s$		$d = 2(s - an)/(n^2 - n).$
15	$a\ l\ s$		$d = (l^2 - a^2)/(2s - l - a).$
16	$n\ l\ s$		$d = 2(nl - s)/(n^2 - n).$
17	$a\ d\ l$	n	$n = (l - a + d)/d.$
18	$a\ d\ s$		$n = [d - 2a \pm \sqrt{(2a-d)^2 + 8ds}]/2d.$
19	$a\ l\ s$		$n = 2s/(a+l).$
20	$d\ l\ s$		$n = [d + 2l \pm \sqrt{(2l+d)^2 - 8ds}]/2d.$

ILLUSTRATIVE PROBLEMS

Find the number of terms in the arithmetic series whose first term is 25, difference -5, and sum 45.

We may substitute in formula 18, but it is quite as easy to use the two fundamental formulas which the student will carry in his mind.

1. From I, $l = 25 + (n-1)(-5) = 30 - 5n.$
2. " II, $45 = \dfrac{25 + 30 - 5n}{2}n.$
3. $\therefore n^2 - 11n + 18 = 0.$
4. $\therefore (n-2)(n-9) = 0$, and $n = 2$, or 9.

The explanation of the two results appears by writing out the series.

$$25,\ 20,\ (15,\ 10,\ 5,\ 0,\ -5,\ -10,\ -15).$$

The part enclosed in parentheses has 0 for its sum.
Hence, the sum of 2 terms is the samé as the sum of 9 terms.

EXERCISE CXXXIV

1. Find s, given $a = 40$, $n = 101$, $d = 5$.
2. Find s, given $a = 1$, $l = 200$, $n = 200$.
3. Find n, given $s = 29{,}000$, $a = 40$, $l = 540$.
4. Find the 200th term in the series 1, 3, 5, ⋯.
5. Insert 7 arithmetic means between -5 and 11.
6. Given $a = -1\frac{1}{3}$ and the 15th term $= 59\frac{1}{3}$, find d.
7. Find the 20th term in the series 540, 480, 420, ⋯.
8. Insert 12 arithmetic means between -18 and 125.
9. Find s, given $a = 14$, $n = 8$, $d = -4$. Write out the series.
10. How many multiples of 17 are there between 350 and 1210?

11. What is the sum of the first 200 numbers divisible by 5? by 7?

12. Show that the sum of any $2n+1$ consecutive integers is divisible by $2n+1$.

13. What is the sum of the first 50 odd numbers? the first 100? the first n?

14. What is the sum of the first 50 even numbers? the first 100? the first n?

15. Given $l=11$, $d=2$, $s=32$, to find n. Check the result by writing out the series.

16. Given $a=15$, $d=-2$, $s=55$, to find n. Check the result by writing out the series.

17. Suppose every term of an arithmetic series to be multiplied by k; is the result an arithmetic series?

18. The sum of four numbers of an arithmetic series is 0 and the sum of their squares is 20. Find the numbers.

19. The sum of four numbers of an arithmetic series is 12 and the sum of their squares is 116. Find the numbers.

20. The sum of three numbers of an arithmetic series is 21 and the sum of their squares is 179. Find the numbers.

21. Find five numbers of an arithmetic series such that the sum of the first and fifth is 46, and that the ratio of the fourth to the second is 1.3.

22. $100 is placed at simple interest annually on the first of each January for 10 yrs., at 6 %. Find the total amount of principals and interest at the end of 10 yrs.

23. Find the nth term and the sum of the first n terms of

(a) $1+\frac{3}{4}+\frac{1}{2}+\cdots$. (b) $11+9+7+\cdots$.

II. GEOMETRIC SERIES

335. Symbols. The following are in common use:

n, s, a, l, as in arithmetic series;

r, the constant by which any term may be multiplied to produce the next; r is usually called the *rate* or *ratio.*

336. Formulas. There are two formulas in geometric series of such importance as to be considered fundamental.

I. $$l = ar^{n-1}.$$

Proof. 1. The second term $= ar$, by definition.

The third term $= ar^2$.

The fourth term $= ar^3$.

$\vdots \qquad \vdots$

2. $\therefore$ the nth term $= ar^{n-1}$.

3. Or $l = ar^{n-1}$.

E.g., the 7th term of the series 16, 8, 4, $\cdots$ is

$$l = 16 \cdot (\tfrac{1}{2})^{7-1} = 16 \cdot \tfrac{1}{64} = \tfrac{1}{4}.$$

II. $$s = \frac{ar^n - a}{r-1} = \frac{lr - a}{r-1}.$$

Proof. 1. $s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$.

2. $\therefore rs = \quad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$, by multiplying by r.

3. $\therefore rs - s = ar^n - a$, by subtracting, (2) − (1).

4. $\therefore (r-1)s = ar^n - a$, and $s = \dfrac{ar^n - a}{r-1}$, by dividing by $(r-1)$.

5. And $\because ar^n = ar^{n-1} \cdot r = lr$, $\therefore s = \dfrac{lr-a}{r-1}$.

E.g., the sum of the first 7 terms of the series 16, 8, 4, $\cdots$, of which l has just been found, is

$$\frac{\frac{1}{4} \cdot \frac{1}{2} - 16}{\frac{1}{2} - 1} = 31\tfrac{3}{4}.$$

337. It is evident that from formulas I and II various others can be deduced.

E.g., given l, a, n, to find r. $\because l = ar^{n-1}$, $\therefore r = (l/a)^{\frac{1}{n-1}}$.

Given n, l, s, to find a. The problem reduces to that of eliminating r from I and II and solving, if possible, for a.

1. From II, $$r = \frac{s-a}{s-l}.$$

2. Substitute this in I, and $$l = a\left(\frac{s-a}{s-l}\right)^{n-1},$$

or $$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$$

Here it is impossible to isolate a. When the numerical values of l, s, n are given, a can frequently be determined by inspection.

For example, given $n = 4$, $l = 8$, $s = 15$, to find a. Here

$$8 \cdot 7^3 = a(15-a)^3,$$

and a evidently equals 8, or 1. Either value checks, for the series may be 8, 4, 2, 1, or 1, 2, 4, 8.

ILLUSTRATIVE PROBLEMS

1. Find the sum of five consecutive powers of 3, beginning with the first.

1. Here $a = 3,\ r = 3,\ n = 5.$
2. $s = (ar^n - a)/(r-1) = (3 \cdot 3^5 - 3)/2 = 363.$

2. Of three numbers of a geometric series, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. Find the numbers.

1. Let x, xy, xy^2 be the numbers.
2. Then $x + xy = xy^2 + 3$, or $x + xy - 3 = xy^2$.
3. And $x + xy^2 = xy + 21$, or $-x + xy + 21 = xy^2$.
4. $\therefore x + xy - 3 = -x + xy + 21$, or $x = 12$.
5. $\therefore 4y^2 - 4y - 3 = 0$, by substituting in 2.
6. $\therefore (2y+1)(2y-3) = 0$, and $y = -\frac{1}{2}$, or $\frac{3}{2}$.
7. $\therefore$ the numbers are 12, -6, 3, or 12, 18, 27. Each set checks.

338. The following table gives the various formulas of geometric series. They should be worked out from formulas I and II by the student, excepting those for n. The formulas for n require logarithms and may be taken after Chap. XX.

	Given	To find	Results
1	$a\ r\ n$		$l = ar^{n-1}$.
2	$a\ r\ s$	l	$l = [a + (r-1)s]/r$.
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
4	$r\ n\ s$		$l = (r-1)sr^{n-1}/(r^n - 1)$.
5	$a\ r\ n$		$s = a(r^n - 1)/(r-1)$.
6	$a\ r\ l$	s	$s = (rl - a)/(r-1)$.
7	$a\ n\ l$		$s = (l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}})/(l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}})$.
8	$r\ n\ l$		$s = l(r^n - 1)/(r^n - r^{n-1})$.
9	$r\ n\ l$		$a = l/r^{n-1}$.
10	$r\ n\ s$	a	$a = s(r-1)/(r^n - 1)$.
11	$r\ l\ s$		$a = rl - (r-1)s$.
12	$n\ l\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
13	$a\ n\ l$		$r = (l/a)^{\frac{1}{n-1}}$.
14	$a\ n\ s$	r	$r^n - sr/a + (s-a)/a = 0$.
15	$a\ l\ s$		$r = (s-a)/(s-l)$.
16	$n\ l\ s$		$r^n - sr^{n-1}/(s-l) + l/(s-l) = 0$.
17	$a\ r\ l$		$n = (\log l - \log a)/\log r + 1$.
18	$a\ r\ s$	n	$n = \{\log[a + (r-1)s] - \log a\}/\log r$.
19	$a\ l\ s$		$n = (\log l - \log a)/[\log(s-a) - \log(s-l)] + 1$.
20	$r\ l\ s$		$n = \{\log l - \log[lr - (r-1)s]\}/\log r + 1$.

EXERCISE CXXXV

1. The sum of how many terms of the series 4, 12, 36, $\cdots$ is 118,096?

2. Find the sum of the first ten terms of the series $3^{\frac{1}{2}}, -2^{\frac{1}{2}}, \frac{2}{3}\cdot 3^{\frac{1}{2}}, \cdots$.

3. Find the geometric means between

   (*a*) 1 and 4. (*b*) -2 and -8.

4. Find the sum of five numbers of a geometric series, the second term being 5 and the fifth 625.

5. What is the fourth term of the geometric series whose first term is 1 and third term $\frac{1}{25}$?

6. The arithmetic mean between two numbers is 39 and the geometric mean 15. Find the numbers.

7. Prove that the geometric mean between two numbers is the square root of their product.

8. Prove that the arithmetic mean between two unequal positive numbers is greater than the geometric mean.

9. To what sum will \$1 amount at 4% compound interest in 5 yrs.? (Here $a = \$1$, $r = 1.04$, $n = 6$.)

10. In Ex. 9, suppose the rate were 4% a year, but the interest compounded semiannually.

11. The sum of the first eight terms of a certain geometric series is 17 times the sum of the first four terms. What is the rate?

Find the 10th term and the sum of the first ten terms of the following series:

12. $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

13. $1, -2, 4, -8, \cdots$.

14. $1, 2, 4, \cdots$.

15. $32, -16, 8, -4, \cdots$.

339. Infinite geometric series. If the number of terms is infinite and $r < 1$, then s approaches as its limit $\dfrac{a}{1-r}$ (§ 141).

This is indicated by the symbols $s \doteq \dfrac{a}{1-r}$, n being infinite.

The symbol $\doteq$ is read "approaches as its limit" (§ 142).

Proof. 1. $\because r < 1$, the terms are becoming smaller, each being multiplied by a fraction to obtain the next.

2. $\therefore l \doteq 0$, and $\therefore lr \doteq 0$, although they never reach that limit.

3. $\therefore s \doteq \dfrac{0-a}{r-1}$, by formula II.

4. $\therefore s \doteq \dfrac{a}{1-r}$, by multiplying each term of the fraction by -1.

E.g., consider the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$, where n is infinite. Here $s \doteq \dfrac{a}{1-r}$, or $\dfrac{1}{1-\frac{1}{2}}$, or 2. That is, the greater the number of terms, the nearer the sum approaches 2, although it never reaches it for finite values of n.

EXERCISE CXXXVI

1. Given $s \doteq 8$, $a = 4$. Find r.
2. Given $s \doteq 10\frac{2}{3}$, $r = \frac{1}{4}$. Find a.
3. Given $s \doteq 1$, $r = \frac{2999}{3000}$. Find a.
4. Given $s = 155$, $r = 2$, $n = 5$. Find a.
5. Given $s = 124.4$, $r = 3$, $n = 4$. Find a.

Find the limits of the following sums, n being infinite:

6. $20 + 10 + 5 + 2\frac{1}{2} + \cdots$.
7. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.
8. $\frac{7}{2} + \frac{7}{8} + \frac{7}{32} + \frac{7}{128} + \cdots$.
9. $10 + 1 + 0.1 + 0.01 + \cdots$.

340. Circulating decimals. If the fraction $\frac{3}{11}$ is reduced to the decimal form, the result is $0.272727 \cdots$, and similarly the fraction $\frac{11}{72} = 0.152777 \cdots$. The former constantly repeats 27, and the latter constantly repeats 7 after 0.152.

When, beginning with a certain order of a decimal fraction, the figures constantly repeat in the same order, the number is called a **circulating decimal**, and the part which repeats is called a **circulate**.

A circulate is represented by a dot over its first and last figures.

$0.272727 \cdots$ is represented by $0.\dot{2}\dot{7}$;
$0.152777 \cdots$ " " " $0.152\dot{7}$.

341. A circulating decimal may be reduced to a common fraction by means of the formula $s \doteq \dfrac{a}{1-r}$, as follows:

1. To what common fraction is $0.\dot{2}\dot{7}$ equal?

1. $0.\dot{2}\dot{7} = 0.27 + 0.0027 + 0.000027 + \cdots$.
2. This is a geometric series with $a = 0.27$, $r = 0.01$, n infinite.
3. $\therefore s \doteq \dfrac{0.27}{1-0.01} = \dfrac{27}{99} = \dfrac{3}{11}$.

2. To what common fraction is $0.152\dot{7}$ equal?

1. $0.152\dot{7} = 0.152 + 0.0007 + 0.00007 + \cdots = 0.152 +$ a geometric series with $a = 0.0007$, $r = 0.1$, n infinite.
2. $\therefore s \doteq \dfrac{0.0007}{1-0.1} = \dfrac{7}{9000}$.
3. To this must be added 0.152, giving $0.152\frac{7}{9}$, or, $\frac{1375}{9000}$, or $\frac{11}{72}$.

EXERCISE CXXXVII

Express as common fractions:

1. $0.\dot{3}$.
2. $0.0\dot{4}\dot{5}$.
3. $0.\dot{0}00\dot{1}$.
4. $0.\dot{1}4\dot{7}$.
5. $1.2\dot{3}7\dot{5}$.
6. $5.\dot{0}50\dot{4}$.
7. $0.\dot{0}4\dot{5}$.
8. $2.0\dot{0}347\dot{1}$.
9. $0.2345\dot{6}$.

III. MISCELLANEOUS TYPES

342. Of the other types of series, some can be treated by the methods which have just been considered.

ILLUSTRATIVE PROBLEMS

1. Defining a harmonic series as one the reciprocals of whose terms form an arithmetic series, insert three harmonic means between 2 and 4.

This reduces to the insertion of three arithmetic means between $\frac{1}{2}$ and $\frac{1}{4}$.

1. $\because\ a = \frac{1}{2}$, $n = 5$, and $l = \frac{1}{4}$,

2. $\therefore\ \frac{1}{4} = \frac{1}{2} + 4d$, and $d = -\frac{1}{16}$.

3. $\therefore$ the arithmetic series is $\frac{1}{2}, \frac{7}{16}, \frac{3}{8}, \frac{5}{16}, \frac{1}{4}$,

and " harmonic " $2, 2\frac{2}{7}, 2\frac{2}{3}, 3\frac{1}{5}, 4$.

2. What is the harmonic mean between a and b?

1. If h is the harmonic mean, $\frac{1}{a}, \frac{1}{h}, \frac{1}{b}$ must form an arithmetic series (Ex. 1).

2. $$\therefore\ \frac{1}{h} - \frac{1}{a} = \frac{1}{b} - \frac{1}{h}.$$

3. $$\therefore\ h = \frac{a+b}{2ab}.$$

E.g., the harmonic mean between 3 and 4 is $\frac{24}{7}$. For, taking the reciprocals of 3, $\frac{24}{7}$, and 4, we have $\frac{1}{3}, \frac{7}{24}, \frac{1}{4}$, or $\frac{8}{24}, \frac{7}{24}$, and $\frac{6}{24}$, which form an arithmetic series.

3. Sum to 20 terms the series, $1, -3, 5, -7, 9, -11, \cdots$.

Here the odd-numbered terms form an arithmetic series with $d = 4$, and the even-numbered ones form an arithmetic series with $d = -4$. There are ten terms in each set. Summing separately, we have

$$190 - 210 = -20.$$

EXERCISE CXXXVIII

1. Sum the series $3, 6, \cdots 3(n-1), 3n$.

2. Sum to $2n$ terms the series $1, -2, +3, -4, \cdots$.

3. Sum the series $1, -3, +5, -7, + \cdots$ to $2n$ terms.

4. Sum the series 15, 13, 10, 7, 5, 1, 0 ⋯ to 20 terms.

5. Sum the series 1, 5, 9, 12, 17, 19, 25, 26, ⋯ to 30 terms.

6. What is the fifth term of the harmonic series 1, 0.5, $0.33\frac{1}{3}$?

7. Sum the series $2, -3, 5, -7, 8, -11, 11, -15, \cdots$ to 40 terms.

8. What is the fourth term of the harmonic series 0.1, 0.125, $0.16\frac{2}{3}$?

9. Insert a harmonic mean between 2 and 2; between -2 and -2.

10. Prove that no two unequal numbers can have their arithmetic, geometric, and harmonic means equal, or any two of these equal.

11. Show that the sum of the first n terms of the series $1, -2, +4, -8, +16, \cdots$ is $\frac{1}{3}(1 \pm 2^n)$, the sign depending on whether n is odd or even.

12. The number of balls in a triangular pile is evidently $1 + (1 + 2) + (1 + 2 + 3) + \cdots$, depending on the number of layers. How many balls in such a pile of 10 layers?

CHAPTER XX

LOGARITHMS

343. About the year 1614 a Scotchman, John Napier, invented a scheme by which multiplication can be performed by addition, division by subtraction, involution by a single multiplication, and evolution by a single division.

344. In considering the annexed series of numbers it is apparent that

$2^0 = 1$	$2^6 = 64$
$2^1 = 2$	$2^7 = 128$
$2^2 = 4$	$2^8 = 256$
$2^3 = 8$	$2^9 = 512$
$2^4 = 16$	$2^{10} = 1024$
$2^5 = 32$	$2^{11} = 2048$

1. $\because 2^3 \cdot 2^5 = 2^8,$

$\therefore 8 \cdot 32 = 2^8 = 256.$

$\therefore$ the product can be found by adding the exponents $(3 + 5 = 8)$ and then finding what 2^8 equals in the annexed table.

2. $\because 2^9 : 2^3 = 2^6,$

$\therefore 512 : 8 = 64.$

$\therefore$ this quotient can be found from the table by a single subtraction of exponents.

3. $\because \quad (2^5)^2 = 2^5 \cdot 2^5 = 2^{10},$

$\therefore \quad 32^2 = 1024.$

4. $\because \quad \sqrt{2^{10}} = \sqrt{2^5 \cdot 2^5} = 2^5,$

$\therefore \quad \sqrt{1024} = 32.$

5. The exponents of 2 form an arithmetic series, while the powers form a geometric series.

In like manner a table of the powers of any number may be made and the four operations, multiplication, division, involution, evolution, reduced to the operations of addition, subtraction, multiplication, and division of exponents.

345. For practical purposes, *the exponents of the powers to which 10, the base of our system of counting, must be raised to produce various numbers* are put in a table, and these exponents are called the **logarithms** of those numbers.

In this connection the word *power* is used in its broadest sense, 10^n being considered as a power, whether n is positive, negative, integral, or fractional. The logarithm of 100 is written "log 100."

E.g., $10^3 = 1000$, $\therefore \log 1000 = 3$. $10^2 = 100$, $\therefore \log 100 = 2$.

$10^0 = 1$, $\therefore \log 1 = 0$. $10^1 = 10$, $\therefore \log 10 = 1$.

$10^{-1} = \frac{1}{10}$, $\therefore \log 0.1 = -1$. $10^{-2} = \frac{1}{10^2}$, $\therefore \log 0.01 = -2$.

$10^{\frac{301}{1000}}$, that is, the thousandth root of 10^{301}, is nearly 2,

$$\therefore \log 2 = 0.301, \text{ nearly.}$$

Although log 2 cannot be expressed exactly as a decimal fraction, it can be found to any required degree of accuracy.

EXERCISE CXXXIX

1. What is the logarithm of 10^{-8}? of 1000^3? of 10^9?

2. What is the logarithm of $10^4 \cdot 10^8$? of $10^7 : 10^3$?

3. What is the logarithm of $\sqrt[3]{10^4} \cdot 10^6 \cdot 10^8$? of $\sqrt[10]{10}$?

4. What is the logarithm of $10^3 \cdot 10^3 \cdot 10^5$? of 0.001 of $10^2 \cdot 10^4$? of $10^3 \cdot 10^5 \cdot 10^9$?

5. Between what two consecutive integers does log 800 lie, and why? also log 3578? log 27?

6. Between what two consecutive negative integers does log 0.02 lie, and why? also log 0.009? log 0.0008?

7. If the logarithm of 2 is 0.301, what is the logarithm of 2^{1000}? $(2 = 10^{\frac{301}{1000}}$, $\therefore$ $2^{1000} = ?$ $\therefore$ the logarithm of $2^{1000} = ?)$

346. Since 2473 lies between 1000 and 10,000, its logarithm lies between 3 and 4. It has been computed to be 3.3932. The integral part 3 is called the *characteristic* of the logarithm, and the fractional part 0.3932 the *mantissa*.

That is, $10^{\frac{33932}{10000}}$, or $10^{3.3932} = 2473$, $\therefore \log 2473 = 3.3932$.

$\because 10^{3.3932} : 10^1 = 10^{2.3932}$, $\therefore 10^{2.3932} = 247.3$, $\therefore \log 247.3 = 2.3932$.

Similarly, $10^{1.3932} = 24.73$, $\therefore \log 24.73 = 1.3932$.

" $10^{0.3932} = 2.473$, $\therefore \log 2.473 = 0.3932$.

" $10^{0.3932-1} = 0.2473$, $\therefore \log 0.2473 = 0.3932 - 1$.

347. It is thus seen that

1. *The characteristic can always be found by inspection.*

Thus, because 438 lies between 100 and 1000, hence log 438 lies between 2 and 3, and $\log 438 = 2 +$ some mantissa.

Similarly, 0.0073 lies between 0.001 and 0.01, hence log 0.0073 lies between -3 and -2, and $\log 0.0073 = -3 +$ some mantissa.

Since 5 lies between 1 and 10, log 5 lies between 0 and 1, and equals $0 +$ some mantissa.

2. *The mantissa is the same for any given succession of digits, wherever the decimal point may be.*

Thus, $\log 2473 = 3.3932$, and $\log 0.2473 = 0.3932 - 1$.

3. *Therefore, only the mantissas need be put in a table.*

Instead of writing the negative characteristic after the mantissa, it is often written before it, but with a minus sign above; thus, $\log 0.2473 = 0.3932 - 1 = \bar{1}.3932$, this meaning that only the characteristic is negative, the mantissa remaining positive.

348. Negative numbers are not considered as having logarithms, but operations involving negative numbers are easily performed. *E.g.*, the multiplication expressed by $1.478 \cdot (-0.007283)$ is performed as if the numbers were positive, and the proper sign is prefixed to the product.

EXERCISE CXL

1. What is the characteristic of the logarithm of a number of 3 integral places? of 6? of 20? of n?

2. What is the characteristic of the logarithm of 0.3? of any decimal fraction whose first significant figure is in the first decimal place? the second decimal place? the 20th? the nth?

3. From Exs. 1, 2 formulate a rule for determining the characteristic of the logarithm of any positive number.

If $\log 39{,}703 = 4.5988$, what are the logarithms of

4. 39,703,000? 5. 397.03? 6. 3.9703?

7. 0.00039703? 8. 0.39703? 9. 3970.3?

349. The fundamental theorems of logarithms.

I. *The logarithm of the product of two numbers equals the sum of their logarithms.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore ab = 10^{m+n}$, and $\log ab = m + n = \log a + \log b$.

Thus, $\log (5 \times 6) = \log 5 + \log 6$.

II. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore \dfrac{a}{b} = \dfrac{10^m}{10^n} = 10^{m-n}$, and $\log \dfrac{a}{b} = m - n = \log a - \log b$.

Thus, $\log (40 \div 5) = \log 40 - \log 5$.

III. *The logarithm of the* n*th power of a number equals* n *times the logarithm of the number.*

1. Let $a = 10^m$, then $\log a = m$.

2. $\therefore a^n = 10^{mn}$, and $\log a^n = nm = n \log a$.

IV. *The logarithm of the* n*th root of a number equals* $\frac{1}{n}$*th of the logarithm of the number.*

1. Let $a = 10^m$, then $\log a = m$.

2. $\therefore a^{\frac{1}{n}} = 10^{\frac{m}{n}}$, and $\log a^{\frac{1}{n}} = \frac{m}{n} = \frac{1}{n} \cdot \log a$.

Th. III might have been stated more generally, so as to include Th. IV, thus: $\log a^{\frac{x}{y}} = \frac{x}{y} \cdot \log a$. The proof would be substantially the same as in Ths. III and IV.

EXERCISE CXLI

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$, and $\log 514 = 2.7110$, find the following:

1. $\log 60$.
2. $\log 24$.
3. $\log 7^8$.
4. $\log \sqrt[3]{2}$.
5. $\log 625$.
6. $\log 7^{\frac{1}{4}}$.
7. $\log \sqrt[5]{3^4}$.
8. $\log \sqrt[3]{21}$.
9. $\log 35$.
10. $\log 514^8$.
11. $\log 1.05$.
12. $\log 257$.
13. $\log 1050$.
14. $\log 154.2$.
15. $\log \sqrt[3]{514}$.
16. $\log 10.28$.
17. $\log 30.84$.
18. $\log 3.598$.
19. $\log 0.3084$.
20. $\log 154{,}200$.
21. $\log 15.42^{\frac{1}{2}}$.
22. $\log 1799 \, [= \log (\frac{1}{2} \cdot 514 \cdot 7)]$.
23. Show how to find $\log 5$, given $\log 2$.

350. Explanation of table. *Given a number to find its logarithm.* In the table on pp. 356 and 357 only the mantissas are given. For example, in the row opposite 71, and under 0, 1, 2, $\cdots$ will be found:

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

This means that the mantissa of log 710 is 0.8513, of log 711 it is 0.8519, and so on to log 719. Hence,

$$\log 715 = 2.8543, \qquad \log 7.18 = 0.8561,$$

$$\log 71{,}600 = 4.8549, \qquad \log 0.0719 = \bar{2}.8567.$$

And $\because$ 7154 is $\frac{4}{10}$ of the way from 7150 to 7160, $\therefore$ log 7154 is about $\frac{4}{10}$ of the way from log 7150 to log 7160.

$\therefore$ log 7154 = log 7150 + $\frac{4}{10}$ of the difference between

log 7150 and log 7160

$= 3.8543 + \frac{4}{10}$ of 0.0006

$= 3.8543 + 0.0002 = 3.8545.$

Similarly, $\log 7.154 = 0.8545,$

and $\log 0.07154 = \bar{2}.8545.$

351. The above process of finding the logarithm of a number of four significant figures is called **interpolation.** It is merely an approximation available within small limits, since numbers do not vary as their logarithms, the numbers forming a geometric series while the logarithms form an arithmetic series. It should be mentioned again that the mantissas given in the table are only approximate, being correct to 0.0001. This is far enough to give a result which is correct to three figures in general, and usually to four, an approximation sufficiently exact for many practical computations.

N	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

352. In all work with logarithms *the characteristic should be written before the table is consulted,* even if it is 0. Otherwise it is liable to be forgotten, in which case the computation will be valueless.

ILLUSTRATIVE PROBLEMS

1. Find from the table log 4260.

The characteristic is 3.

The mantissa is found to the right of 42 and under 6; it is 0.6294.

$\therefore \log 4260 = 3.6294.$

2. Find from the table log 42.67.

The characteristic is 1.

$$\begin{aligned} \log 42.7 &= 1.6304 \\ \log 42.6 &= 1.6294 \\ \text{difference} &= 0.0010 \\ \tfrac{7}{10} \text{ of } 0.0010 &= 0.0007 \\ \therefore \log 42.67 &= 1.6294 + 0.0007 \\ &= 1.6301. \end{aligned}$$

EXERCISE CXLII

From the table find the following:

1. log 28.	2. log 443.	3. log 9.823.
4. log 2.34.	5. log 6.81.	6. log 700.3.
7. log 8940.	8. log 43.41.	9. $\log \sqrt[4]{125}$.
10. log 3855.	11. log 2.005.	12. $\log 9.821^5$.
13. log 1003.	14. log 3.142.	15. log 24,000.
16. log 23.42.	17. $\log \sqrt{4.28}$.	18. log 0.2346.
19. $\log 75.55^{\frac{1}{3}}$.	20. log 0.0007.	21. log 0.00323.
22. log 0.2969.	23. $\log 0.0129^3$.	24. log 0.000082.

353. *Given the logarithm to find the corresponding number.* The number to which a logarithm corresponds is called its **antilogarithm**.

E.g., $\because \log 2 = 0.3010, \therefore \text{antilog } 0.3010 = 2.$

The method of finding antilogarithms will be seen from a few illustrations. Referring again to the row after 71 on p. 357, we have

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

Hence, we see that

$\text{antilog } 0.8513 = 7.1,$ $\text{antilog } 5.8531 = 713{,}000,$

$\text{antilog } \bar{2}.8567 = 0.0719,$ $\text{antilog } \bar{1}.8555 = 0.717.$

Furthermore, $\because$ 8540 is halfway from 8537 to 8543,

$\therefore$ antilog 2.8540 is about halfway from antilog 2.8537 to antilog 2.8543.

$\therefore$ antilog 2.8540 is about halfway from 714 to 715.

$$\therefore \text{antilog } 2.8540 = 714.5.$$

Similarly, to find antilog $\bar{1}.8563$.

$$\begin{array}{lr} \text{antilog } \bar{1}.8567 = 0.719 & \bar{1}.8563 \\ \text{antilog } \underline{\bar{1}.8561} = 0.718 & \underline{\bar{1}.8561} \\ \qquad\qquad 6 & 2 \end{array}$$

$$\therefore \text{antilog } \bar{1}.8563 = 0.718\tfrac{2}{6} = 0.7183.$$

354. The interpolation here explained is, as stated on p. 355, merely a close approximation; it cannot be depended upon to give a result beyond four significant figures except when larger tables are employed.

This is sufficient in many numerical computations. *E.g.*, we speak of the distance to the sun as 93,000,000 mi., using only two significant figures.

EXERCISE CXLIII

From the table find the following:

1. antilog 0.3234.
2. antilog 2.4271.
3. antilog $\bar{2}.9193$.
4. antilog 5.2183.
5. antilog 3.9286.
6. antilog $\bar{1}.7929$.
7. antilog 0.8996.
8. antilog 4.7834.
9. antilog 3.9320.
10. antilog 2.0000.
11. antilog 1.9850.
12. antilog 0.7076.
13. antilog 10.5445.
14. antilog 3.6987.
15. antilog $0.9485 - 4$.
16. antilog $0.6585 - 6$.
17. antilog $0.6120 - 2$.
18. antilog $0.9290 - 3$.

355. Cologarithms. In cases of division by a number n it is often more convenient to add the logarithm of $\frac{1}{n}$ than to subtract the logarithm of n. The logarithm of $\frac{1}{n}$ is called the **cologarithm** of n.

$$\because \log\frac{1}{n} = \log 1 - \log n = 0 - \log n,$$

$$\therefore \operatorname{colog} n = - \log n.$$

Also, $\operatorname{colog} n = 10 - \log n - 10$, often a more convenient form to use.

E.g., $\because \log 6 = 0.7782.$

$\therefore \operatorname{colog} 6 = - 0.7782.$

This may also be written $10 - 0.7782 - 10$, or $9.2218 - 10$.

The object of this is seen when we consider the addition of several logarithms and cologarithms; it is easier to add if all the mantissas are positive, subtracting the 10's afterward.

In general, $\operatorname{colog} n = 10p - \log n - 10p$; that is, we may use 10, 20, or any multiple of 10, as may be most convenient.

356. The cologarithm can evidently be found by mentally subtracting each digit from 9, excepting the right-hand significant one (which must be subtracted from 10) and the zeros following, and then subtracting 10.

E.g., to find colog 6178.

$$\begin{array}{r@{\;}l}
 & 9.\ \ 9\ \ 9\ \ 9\ 10 \\
\log 6178 = & \underline{3.\ \ 7\ \ 9\ \ 0\ \ 9} \\
\operatorname{colog} 6178 = & 6.\ \ 2\ \ 0\ \ 9\ \ 1 - 10.
\end{array}$$

To find colog 41.5.

$$\begin{array}{r@{\;}l}
 & 9.\ \ 9\ \ 9\ 10\ \ 0 \\
\log 41.5 = & \underline{1.\ \ 6\ \ 1\ \ 8\ \ 0} \\
\operatorname{colog} 41.5 = & 8.\ \ 3\ \ 8\ \ 2\ \ 0 - 10.
\end{array}$$

To find colog 0.013.

$$\begin{array}{r@{\;}l}
 & 9.\ \ 9\ \ 9\ \ 9\ 10 \\
\log 0.013 = & \underline{\bar{2}.\ \ 1\ \ 1\ \ 3\ \ 9} \\
\operatorname{colog} 0.013 = & 11.\ \ 8\ \ 8\ \ 6\ \ 1 - 10 = 1.8861.
\end{array}$$

357. In case the characteristic exceeds 10 but is less than 20, colog n may be written $20 - \log n - 20$, and so for other cases; but these cases are so rare that they may be neglected at this time.

The advantage of using cologarithms will be apparent from a single example:

To find the value of $\dfrac{317 \cdot 92}{6178 \cdot 0.13}$.

Using Cologarithms

$$\begin{array}{r@{\;}l}
\log 317 = & 2.5011 \\
\log 92 = & 1.9638 \\
\operatorname{colog} 6178 = & 6.2091 - 10 \\
\operatorname{colog} 0.13 = & \underline{10.8861 - 10} \\
\log 36.32 = & 1.5601
\end{array}$$

$$\therefore \frac{317 \cdot 92}{6178 \cdot 0.13} = 36.32.$$

Not using Cologarithms

$$\begin{array}{r@{\;}l}
\log 317 = & 2.5011 \\
\log 92 = & \underline{1.9638} \\
\underline{\log (317 \cdot 92) =} & \underline{4.4649} \\
\log 6178 = & 3.7909 \\
\log 0.13 = & \underline{\bar{1}.1139} \\
\underline{\log (6178 \cdot 0.13) =} & \underline{2.9048} \\
\log (317 \cdot 92) = & 4.4649 \\
\log (6178 \cdot 0.13) = & \underline{2.9048} \\
\log 36.32 = & 1.5601
\end{array}$$

358. Various bases. Thus far we have considered logarithms as exponents of powers of 10. But it is evident that various other *bases* might be taken. Logarithms *to the base 10,* such as we have thus far considered, are sometimes called *common* or *Briggs logarithms,* the latter designation being in honor of Henry Briggs, who is said to have suggested this base to Napier.

If 2 were the base, log 8 would be 3, because $2^3 = 8$. Similarly, log 16 would be 4, and so on.

Where a different base than 10 is used (which is not the case in practical calculations), or where more than one base is used in the same discussion, the base is indicated by a subscript; thus $\log_2 32 = 5$, because $2^5 = 32$.

359. Computations by logarithms. A few illustrative problems will now be given. It is urged that all work be neatly arranged, since as many errors arise from failure in this respect as from any other single cause.

1. Find the value of $\dfrac{0.007^3}{0.03625}$.

$$\begin{aligned}
\log 0.007 &= \underline{0.8451 - 3} \\
3 \cdot \log 0.007 &= 2.5353 - 9 \\
\operatorname{colog} 0.03625 &= \underline{11.4407 - 10} \\
&\ \ 13.9760 - 19 \\
&= 0.9760 - 6 = \log 0.000009462.
\end{aligned}$$

$\therefore 9.462 \cdot 10^{-6} = Ans.$

It will be noticed that the negative characteristic is less confusing if written by itself at the right.

2. Find the value of $0.09515^{\frac{1}{3}}$.

$$\log 0.09515 = 0.9784 - 2.$$

$\because$ the characteristic (-2) is not divisible by 3, this may be written

$$\log 0.09515 = 1.9784 - 3.$$

Then $$\tfrac{1}{3} \log 0.09515 = 0.6595 - 1 = \log 0.4566.$$

$\therefore 0.4566 = Ans.$

3. Given a, r, l, in a geometric series, to find n. Compute the value if $l = 256$, $a = 1$, $r = 2$.

1. From § 336, $$l = ar^{n-1}.$$

2. $$\therefore \log l = \log a + (n-1)\log r. \qquad \text{§ 349}$$

3. $$\therefore \frac{\log l - \log a}{\log r} + 1 = n.$$

$$\log 256 = 2.4082;$$

$$\log 1 = 0, \log 2 = 0.3010;$$

$$2.4082 \div 0.3010 = 8.$$

4. $$\therefore n = 8 + 1 = 9.$$

4. Find the value of $\dfrac{2.706 \cdot 0.3 \cdot 0.001279}{86{,}090}$.

This may at once be written $\dfrac{2.706 \cdot 3 \cdot 1.279}{8.609} \cdot 10^{-8}$, thus simplifying the characteristics. Then

$$\begin{aligned} \log 2.706 &= 0.4324 \\ \log 3 &= 0.4771 \\ \log 1.279 &= 0.1069 \\ \text{colog } 8.609 &= \underline{9.0650 - 10} \\ \log 1.206 &= 0.0814 \end{aligned}$$

$\therefore 1.206 \cdot 10^{-8} = Ans.$

5. Given $2^x = 7$, find x, the result to be correct to 0.01.

$$x \log 2 = \log 7.$$

$$\therefore x = \frac{\log 7}{\log 2} = \frac{0.8451}{0.3010} = 2.81.$$

This division might be performed by finding the antilogarithm of $(\log 0.8451 - \log 0.3010)$, a plan not expeditious in this case.

EXERCISE CXLIV

In the following exercises give the results to four significant figures:

1. Find the value of $37^{\frac{1}{37}}$.

2. Find the value of $\sqrt[100]{100}$.

3. Find the value of $(32/29)^{\frac{4}{5}}$.

4. Find the value of $(37/2939)^{1\frac{2}{3}}$.

5. Find the value of $\sqrt[10]{2\sqrt[10]{2} : \sqrt{10}}$.

6. Find the value of $(3.64/7.985)^{6}$.

7. Find the value of $(1402/3999)^{-3.8}$.

8. Find the value of $(22.8 \div 0.09235)^{\frac{1}{3}}$.

9. Find the value of $(24.73^{5} \div 31.97^{4})^{\frac{1}{5}}$.

10. Find the value of $(44 \cdot 8.37)^{\frac{1}{3}} \div 0.227^{\frac{1}{2}}$.

11. Given $x : 5.127 = 0.325 : 2936$. Find x.

12. Given a, r, s, in a geometric series, show that

$$n = \frac{\log[a + (r-1)s] - \log a}{\log r},$$

and compute the value of n when $a = 1$, $r = 2$, $s = 511$.

13. Also, given r, l, s, show that

$$n = \frac{\log l - \log[lr - (r-1)s]}{\log r} + 1.$$

Compute the value of n when $r = 3$, $l = 729$, $s = 1092$.

14. Also, given a, l, s, show that

$$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1.$$

Compute the value of n when $a = 3$, $l = 729$, $s = 1092$.

15. Solve the equation $5^{x} = 6$. (First take the logarithm of each member.)

CHAPTER XXI

PERMUTATIONS AND COMBINATIONS

360. The different groups of 2 things that can be selected from a collection of 3 different things, without reference to their arrangement, are called the **combinations** of 3 things taken 2 at a time.

E.g., representing the 3 things by the letters a, b, c, we can select 2 things in 3 ways, ab, ac, bc.

In general, the different groups of r things which can be selected from a collection of n different things, without reference to their arrangement, are called the **combinations of n things taken r at a time.**

So the combinations of the 4 letters a, b, c, d, taken 3 at a time, are abc, abd, acd, bcd; taken 2 at a time, ab, ac, ad, bc, bd, cd.

EXERCISE CXLV

1. What is the number of combinations of 5 things taken 2 at a time? Represent them by letters.

2. What is the number of combinations of 5 things taken 3 at a time? Represent them by letters.

3. Write out the combinations of the letters w, x, y, z, taken 4 at a time; 3 at a time; 2 at a time; 1 at a time.

4. How does the number of combinations of 6 things taken 2 at a time compare with the number taken 4 at a time?

361. The different groups of 2 things which can be selected from 3 things, varying the arrangements in every possible manner, are called the **permutations** of 3 things taken 2 at a time.

E.g., the permutations of the letters a, b, c, taken 2 at a time, are ab, ba, ac, ca, bc, cb.

362. In general, the different groups of r things which can be selected from n different things, varying the arrangements in every possible manner, are called the **permutations of n things taken r at a time.**

In all this work the things are supposed to be different, and not to be repeated, unless the contrary is stated.

363. The number of combinations of n things taken r at a time is indicated by the symbol C^n_r. The number of permutations of n things taken r at a time is indicated by the symbol P^n_r.

EXERCISE CXLVI

1. Show that $P^4_2 = 12$.

2. Show that $P^4_3 = 2 \cdot P^4_2$.

3. Show that $P^4_2 = 2 \cdot C^4_2$.

4. Find the value of P^5_2; of P^5_3.

5. Show that $C^n_1 = n$, and $C^n_n = 1$.

6. Show that $P^3_1 = 3$, and in general that $P^n_1 = n$.

7. Using the letters a, b, c, show that $C^3_2 = 3$.

8. Write out the permutations of the letters of the word *time*, taken all together.

9. Write out the permutations of the letters a, b, c, d taken 2 at a time; 3 at a time.

364. Theorem. *The number of permutations of* n *different things taken* r *at a time is* $n(n-1)(n-2)\cdots(n-r+1)$.

Proof. 1. Since we are to take r things, we may suppose there are r places to be filled.
The first place may be filled in any one of n ways.

Thus, with a, b, c, d, we may fill the first place with a, b, c, or d.

2. For every way of filling the first place there are $n-1$ ways of filling the first and second.

Thus, if the first place be filled with a, we may fill the first and second with ab, ac, ad.

3. $\therefore$ for n ways of filling the first place there are $n(n-1)$ ways of filling the first two.

E.g.,

$$\begin{array}{ccc} ab, & ac, & ad, \\ ba, & bc, & bd, \\ ca, & cb, & cd, \\ da, & db, & dc, \end{array}$$

giving $4 \cdot 3 = 12$ ways in all.

4. For every way of filling the first two places there are $n-2$ ways of filling the first, second, and third.

Thus, if the first 2 places be filled with ab, the first 3 can be filled with abc, abd, *i.e.*, in $4-2$ ways.

5. $\therefore$ for $n(n-1)$ ways of filling the first two places there are $n(n-1)(n-2)$ ways of filling the first three.

E.g.,

$$\begin{array}{cccc} abc, & abd, & adc, & adb, \\ acb, & acd, & bca, & bcd, \\ bda, & bdc, & cda, & cdb, \end{array}$$

and the same with the first two letters interchanged in each.

6. Similarly, the number taken 4 at a time is $n(n-1)(n-2)(n-3)$, and the same reasoning evidently shows that the number of permutations of n things r at a time is

$$n(n-1)(n-2)\cdots(n-\overline{r-1})$$

or $$n(n-1)(n-2)\cdots(n-r+1).$$

365. Corollary. *If* $n=r$, $P^n_n = n(n-1)\cdots 3\cdot 2\cdot 1$. *Hence, the number of permutations of* n *things taken all together is* $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

EXERCISE CXLVII

1. Find the value of P^{100}_2.

2. Find the value of P^{20}_4.

3. Prove that $P^{n-1}_{n-1} = \frac{1}{n}P^n_n$.

4. Prove that $P^n_n = P^n_r \cdot P^{n-r}_{n-r}$.

5. Find the value of P^5_2; of P^6_3. Prove this by writing out the permutations of the letters $a, b, c, \cdots$.

6. Show from the theorem (§ 364) that P^n_r is greater as r is greater.

7. Show from the corollary that

$$P^n_n = nP^{n-1}_n = (n^2-n)P^{n-2}_n.$$

8. Find the number of permutations of the letters of the word *number* taken all together.

9. Find the number of permutations of the letters of the word *courage* taken 3 at a time; taken all together.

10. By writing out the permutations and the combinations of the letters a, b, c, d, e, taken 2 at a time, ascertain how P^5_2 compares with C^5_2.

366. Factorials. The product

$$n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1,$$

that is, of all integers from 1 to n inclusive, is called **factorial n.**

Thus, factorial $3 = 1\cdot 2\cdot 3 = 6,$

" $4 = 1\cdot 2\cdot 3\cdot 4 = 24$, etc.

Factorial n is represented by several symbols. In writing it is customary to use $\underline{|n}$, this being a symbol easily made. In print, on account of the difficulty of setting the $\underline{|n}$, it is customary to use the symbol $n!$

We shall use in print only the symbol $n!$

367. It therefore appears that

(1) $P^n_n = n!$

(2) $P^n_r = \dfrac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1} = \dfrac{n!}{(n-r)!}$

EXERCISE CXLVIII

1. Show that $P^{10}_4 = \dfrac{10!}{6!}$.

2. Show that $5! = 120$.

3. Find the value of $\dfrac{10!}{5!}$.

4. Also of $\dfrac{12!}{10!}\cdot\dfrac{8!}{6!}\cdot\dfrac{4!}{2!}$.

5. Prove that $P^n_r : P^n_{n-r} = r! : (n-r)!$

6. Prove that $n! = n(n-1)(n-2)\cdot(n-3)!$

7. Prove that $(n!)^2 = n^2(n-1)^2(n-2)^2\cdots 3^2\cdot 2^2\cdot 1$.

8. In how many ways can 10 persons be placed in a row?

9. How many permutations of 7 different things, taken all together, are possible?

10. How many arrangements (permutations) can be made of the letters a, b, c, d, e, taken all together in a line?

368. Theorem. *The number of permutations of* n *different things taken* r *at a time, when each of the* n *things may be repeated, is* n^r.

Proof. After the first place has been filled, the second can be filled in n ways, since repetition is allowed. So for the subsequent places.

Hence, instead of having

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1),$$

we have $$n \cdot n \cdot n \cdots n = n^r.$$

EXERCISE CXLIX

1. Find the value of P_4^6, repetitions being allowed.

2. Find the value of P_5^5, repetitions being allowed.

3. How many numbers are there containing 4 digits?

4. How many ways are there of selecting 3 numbers from 50 on a combination lock, repetitions being allowed?

5. How many ways are there of selecting 3 numbers from 10 on a combination lock, repetitions being allowed?

6. Show that P_n^n, repetitions being allowed, is n^n. From this tell how many 9-figure numbers are possible, all zeros being excluded.

7. From Ex. 6, how many 10-figure numbers are possible, zeros being admitted except in the highest order?

8. How many possible integral numbers can be formed from the digits 1, 2, 3, 4, or any of them, repetitions of the digits being allowed?

9. The chance of guessing correctly, the first time, the three numbers on which a combination lock of 100 numbers is set, is 1 out of how many?

369. Theorem. *The number of combinations of* n *different things taken* r *at a time is*

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Proof. 1. For each combination of r things there are $r!$ permutations.

2. $\therefore$ for C^n_r combinations there are $C^n_r \times r!$ permutations.

3. But it has been shown that this number of permutations is

$$n(n-1)(n-2)\cdots(n-r+1). \qquad \S\ 364$$

4. $\therefore$ $C^n_r \times r! = n(n-1)(n-2)\cdots(n-r+1),$

and $$C^n_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

370. Corollaries. 1. $C^n_r = P^n_r/r!$

2. $C^n_r = \dfrac{n!}{r!(n-r)!}$.

For we may multiply both terms of the fraction

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

by $(n-r)!$, giving

$$\frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!(n-r)!}$$

which equals $$\frac{n!}{r!(n-r)!}$$

This is a more convenient formula to write and to carry in mind. Practically, of course, it gives the same result as the other. *E.g.*,

By the theorem, $$C^5_3 = \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1};$$

by the corollary, $$C^5_3 = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1}.$$

EXERCISE CL

1. If $P_n^n = 3{,}628{,}800$, find n.

2. Find the values of P_3^7; of P_5^{10}; of C_3^8.

3. If $P_2^n = 56$, find n, and explain why there should be two results.

4. In how many ways can 3 persons be selected from a class of 20?

5. In how many ways can the letters of the word *cat* be arranged?

6. Prove that $C_r^n = C_{n-r}^n$, by substituting in the formula of § 370, cor. 2.

7. What is the number of combinations of 20 things taken 5 at a time?

8. In how many ways can the letters of the word *combine* be arranged?

9. How many numbers can be formed by taking 4 out of the 5 digits 1, 2, 3, 4, 5?

10. How many triangles are formed from 4 lines, each of which intersects the other 3?

11. How many changes can be rung with a peal of 7 bells, a particular one always being last?

12. In how many ways may the letters of the word *united* be arranged, taken all at a time?

13. How many changes can be rung with a peal of 5 bells, using each bell once in each change?

14. In how many ways can a consonant and a vowel be chosen out of the letters of the word *numbers*?

15. How many numbers between 200 and 300 have [illegible]?

16. In how many ways may the letters of the word [illegible] be arranged, taken any number at a time?

17. In how many ways can 5 persons be seated about a circular table, one of them always occupying the same place?

18. How many different arrangements (permutations) can be made by taking 5 of the letters of the word *rectangle*?

19. On an examination 15 questions are given, of which the student has a choice of 10. In how many ways may he make his selection?

20. How many different arrangements can be made of the letters of the word *algebra*, it being noted that two of the letters are alike?

21. There are four points in a plane, no three being in the same straight line. How many straight lines can be drawn connecting two points?

22. How many different signals can be made with 5 different flags, displayed on a staff 3 at a time? 4 at a time? 2 at a time? all together? any number at a time?

23. Suppose a telegraphic system consists of two signs, a dot and a dash; how many letters can be represented by these signs taken 1 at a time? 2 at a time? 3 at a time? 4 at a time?

24. Prove that the number of permutations of n different things taken r at a time is $n-r+1$ times the number of permutations of the n things taken $r-1$ at a time.

CHAPTER XXII

THE BINOMIAL THEOREM

371. The binomial theorem is stated in § 184, but without proof.

It is now proposed to consider this theorem in the light of Chapter XXI.

372. Theorem. *If the binomial* a + b *is raised to the* n*th power,* n *integral and positive, the result is expressed by the formula*

$$(\mathrm{x} + \mathrm{a})^{\mathrm{n}} = \mathrm{x}^{\mathrm{n}} + C_1^n \mathrm{x}^{\mathrm{n}-1}\mathrm{a} + C_2^n \mathrm{x}^{\mathrm{n}-2}\mathrm{a}^2$$
$$+ C_3^n \mathrm{x}^{\mathrm{n}-3}\mathrm{a}^3 + \cdots C_{n-1}^n \mathrm{x}\mathrm{a}^{\mathrm{n}-1} + \mathrm{a}^{\mathrm{n}}.$$

Proof. 1. By multiplication we know that

$$(x + a)(x + b)$$
$$= x^2 + (a + b)x + ab,$$

$$(x + a)(x + b)(x + c)$$
$$= x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc,$$

$$(x + a)(x + b)(x + c)(x + d)$$
$$= x^4 + (a + b + c + d)x^3$$
$$+ (ab + ac + ad + bc + bd + cd)x^2$$
$$+ (abc + abd + acd + bcd)x + abcd.$$

There is evidently a law running through all these expansions, relating to the exponents and the coefficients of x.

2. We might infer from step 1 that if there were n factors, the product would have for the coefficient

   of x^n, 1;

   of x^{n-1}, $a + b + c \cdots n$;

   of x^{n-2}, the combinations of the letters $a, b, \cdots n$, taken 2 at a time;

   of x^{n-3}, the combinations of these letters taken 3 at a time;

   ⋮

   of x, the combinations of these letters taken $n-1$ at a time.

3. This inference is correct; for the term containing x^n can be formed only by taking the product of the x's in all the factors, and hence its coefficient is 1.
   The terms containing x^{n-1} can be formed only by multiplying the x's in all but one factor by the other letter in that factor; hence the x^{n-1} term will have for its coefficient $(a + b + \cdots n)$.
   The terms containing x^{n-2} can be formed only by multiplying the x's in all but 2 factors by the other letters in those factors, *i.e.*, by a and b, a and c, a and d, etc.; hence the x^{n-2} term will have for its coefficient $(ab + ac + ad + \cdots)$.
   The reasoning is evidently general for the rest of the coefficients.

4. If, now, we let $a = b = c = \cdots = n$, we have

$$(x+a)^n = x^n + C_1^n x^{n-1}a + C_2^n x^{n-2}a^2 + C_3^n x^{n-3}a^3 + \cdots + C_{n-1}^n x a^{n-1} + a^n.$$

373. It can be proved that this is true even if n is negative or fractional. The proof is, however, too difficult for the student at this time.

Assuming that the theorem is true whether n is positive or negative, integral or fractional, it offers a valuable exercise in the use of negative and fractional exponents.

ILLUSTRATIVE PROBLEMS

1. Required the square root of $1 + x$ to 3 terms.

1. $\because \quad (a+b)^n = a^n + na^{n-1}b + \dfrac{n(n-1)}{2}a^{n-2}b^2 + \cdots,$

2. $\therefore \quad (1+x)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \frac{1}{2}\cdot 1^{-\frac{1}{2}}\cdot x + \dfrac{\frac{1}{2}(\frac{1}{2}-1)}{2}\cdot 1^{-\frac{3}{2}}\cdot x^2 + \cdots$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots.$$

2. Expand to 4 terms $(a - 2b)^{-3}$.

1. $\because \quad (x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{2}x^{n-2}y^2$

$$+ \frac{n(n-1)(n-2)}{2\cdot 3}x^{n-3}y^3 + \cdots,$$

2. $\therefore \quad (a-2b)^{-3} = a^{-3} + (-3)\,a^{-4}(-2b)$

$$+ \frac{-3\cdot -4}{2}a^{-5}(-2b)^2$$

$$+ \frac{-3\cdot -4\cdot -5}{2\cdot 3}a^{-6}(-2b)^3 + \cdots$$

$$= a^{-3} + 6a^{-4}b + 24a^{-5}b^2 + 80a^{-6}b^3 + \cdots.$$

3. Expand to 3 terms $(1+x)^{-\frac{1}{4}}$.

As above, $(1+x)^{-\frac{1}{4}} = 1 + (-\frac{1}{4})x + \dfrac{(-\frac{1}{4})(-\frac{1}{4}-1)}{2}x^2 + \cdots$

$$= 1 - \tfrac{1}{4}x + \tfrac{5}{32}x^2 \cdots.$$

4. Extract the square root of 5.

$$\sqrt{5} = (4+1)^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}} + \tfrac{1}{2}\cdot 4^{-\frac{1}{2}} - \tfrac{1}{8}\cdot 4^{-\frac{3}{2}} + \tfrac{1}{16}\cdot 4^{-\frac{5}{2}} - \cdots$$

$$= 2 + \tfrac{1}{4} \qquad - \tfrac{1}{64} \qquad + \tfrac{1}{512} \qquad - \cdots.$$

EXERCISE CLI

Expand to four terms expressions 1–10.

1. $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$.

2. $\sqrt{14} = \sqrt{16-2} = 4\left(1-\frac{1}{8}\right)^{\frac{1}{2}}$.

3. $(1+x)^{\frac{1}{4}}$.

4. $(1+x)^{\frac{1}{2}}$.

5. $(1+x)^{-1}$.

6. $1/\sqrt{1-x}$.

7. $(1+x)^{-\frac{1}{4}}$.

8. $(1-2a)^{\frac{3}{4}}$.

9. $(1+x)^{-\frac{2}{3}}$.

10. $(3x-2y)^{\frac{2}{3}}$.

Find the 5th term in the expansion of expressions 11–14.

11. $(1+x)^{\frac{2}{3}}$.

12. $(1-x)^{\frac{3}{4}}$.

13. $(1-x)^{-2}$.

14. $(1-x)^{-4}$.

Find by expanding the indicated binomial to four terms, and reducing these to decimal fractions and adding, the following roots:

15. $\sqrt{10} = (9+1)^{\frac{1}{2}}$.

16. $\sqrt{82} = (81+1)^{\frac{1}{2}}$.

17. $\sqrt[3]{28} = (27+1)^{\frac{1}{3}}$.

18. $\sqrt{37} = (36+1)^{\frac{1}{2}}$.

19. $\sqrt{24} = (25-1)^{\frac{1}{2}}$.

20. $\sqrt{50} = (49+1)^{\frac{1}{2}}$.

Expand to five terms expressions 21–23.

21. $(1-x)^{-1}$. Check by performing the division $\dfrac{1}{1-x}$.

22. $(1-x)^{-2}$. Check by performing the division

$$\frac{1}{1-2x+x^2}.$$

23. $(1+x)^{-2}$. Check by performing the division

$$\frac{1}{1+2x+x^2}.$$

REVIEW EXERCISE CLII

The following eight groups of exercises serve as typical examination papers covering the work of algebra through quadratic equations.

A

1. Solve the system $x^3 + y^3 = 729,\ xy = 0.$

2. Simplify the expression $\dfrac{x^2 + y^2}{x - y} - \dfrac{x^3 - y^3}{x^2 + y^2}.$

3. Solve the system $\dfrac{3x}{a} - \dfrac{2y}{b} = 9,\ \dfrac{2x}{a} - \dfrac{3y}{b} = 1.$

4. Find the l. c. m of $x^4 + x^2 + 1$ and $x^4 - x^2 + 1.$

5. Simplify the expression $(\sqrt{-1})^{4n} + (\sqrt{-1})^{6n}.$

6. Factor the expression $x^4 + 105 + 2x^2(4x - 5) - 104x.$

7. Solve the system

$x + y + z = 10,\ 3x + y + 2z = 22,\ 5x - 3y + 7z = 72.$

8. Form the quadratic equation whose roots are $a^{\frac{1}{2}}b^{\frac{2}{3}}$ and $\sqrt{a^{\frac{3}{2}}b^{\frac{1}{3}}}.$

9. Find the h. c. f. of $x^4 - 3x^3 + 2x^2 + 2x - 1$ and $x^3 - x^2 - 2x + 2.$

10. A man invests \$ 7200 in a 4% stock, and an equal sum in a 3% stock. The income from the first investment exceeds that from the second by \$ 50. If the price of each stock had been \$ 10 higher, the difference between the incomes would have been \$ 48. Required the prices of the two stocks.

B

1. Factor the expression $x^2(x^2-3)-x(3x^2-7)+6$.

2. Find the h. c. f. of
$a^4-7a^3+8a^2+28a-48$ and $a^3-8a^2+19a-14$.

3. Find the l. c. m. of
a^2-2a-3, a^3+a^2-4a-4, and a^3-7a-6.

4. Simplify the expression $\dfrac{(a-3)^2}{(a+3)^2}-\dfrac{a-3}{a+3}$.

5. Solve the system $\dfrac{6x}{a^2}+\dfrac{y}{b^2}=23$, $\dfrac{2x}{a^2}+\dfrac{3y}{b^2}=21$.

6. Solve the system $2x-2y+z=2$, $3x+3y-z=10$, $x+5y+3z=18$.

7. Square the complex number $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

8. Form the quadratic equation whose roots are $1\pm\sqrt{-1}$.

9. Solve the system $y^2-x^2=8$, $y-x=1$.

10. A boat's crew row 3.5 mi. down a river and back again in 1 hr. and 40 mins. Supposing the river to have a current of 2 mi. per hour, find the rate at which the crew would row in still water.

C

1. Factor the expression $(a^2+2-3a)(a^2+3a+2)$.

2. Find the h. c. f. of
$p^4-p^3+2p^2+p+3$ and p^4+2p^3-p-2.

3. Find the l. c. m. of a^6-8, a^6+8, a^8+4a^4+16.

4. Simplify the expression $\dfrac{x^2+x+1}{x^2-x+1}+\dfrac{x+1}{x-1}$.

5. Solve the system $\dfrac{x}{p+q}+\dfrac{y}{p-q}=1$, $\dfrac{x}{p+q}-\dfrac{y}{p-q}=1$.

6. Solve the system
$13x - 7y + z = 8,\ 3x + 4y + 3z = 13,\ 5x - 2y + 7z = 17.$

7. Square the complex number $\dfrac{1-\sqrt{-1}}{\sqrt{2}}$.

8. Show that the roots of $x^2 + px + q = 0$ will be rational if $p = k + \dfrac{q}{k}$, where p, q, k are rational quantities.

9. Solve the system $x^2 + y^2 = 113,\ x + y = 15$.

10. A man rents an orchard for \$ 84. He reserves 4 acres for himself and lets the rest for 50 cents an acre more than he pays for it. He thereby receives for this portion the whole rent, \$84. Required the number of acres.

D

1. Factor the expression $a^4 - ma^3 + (n-1)a^2 + ma - n$.

2. Find the h. c. f. of $4m^4 + 9m^3 + 2m^2 - 2m - 4$ and $3m^3 + 5m^2 - m + 2$.

3. Find the l. c. m. of $m^6 - n^6$, $m^9 + n^9$, $m^8 + m^4n^4 + n^8$.

4. Simplify the expression $\dfrac{1}{x+2a} + \dfrac{1}{x-2a} + \dfrac{8a^2}{4a^2x - x^3}$.

5. Solve the system $p^3x + q^3y = p^3q^3,\ p^4x + q^4y = p^4q^4$.

6. Solve the system
$3x + y - z = 16,\ 4x + 2y + 3z = 70,\ z - x - y = -4.$

7. Square the complex number $\dfrac{1+\sqrt{-1}}{\sqrt{2}}$.

8. Solve the equation $x^{\frac{2}{a}} - \frac{5}{2}x^{\frac{1}{a}} + 1 = 0$.

9. Solve the system $\dfrac{x}{y} + \dfrac{y}{x} - \dfrac{5}{2} = 0,\ xy = 8$.

10. Find two numbers whose sum is 9 times their difference, and whose product diminished by the greater number is equal to 12 times the greater number divided by the less.

E

1. Factor the expression a^2+2b-b^2-1.

2. Find the h.c.f. of $9-6a+19a^2-12a^3+2a^4$ and $4a^3-18a^2+19a-3$.

3. Find the l.c.m. of a^2-b^2, a^3-b^3, a^4-b^4, a^5-b^5, a^6-b^6.

4. Simplify the expression $\dfrac{a^4}{b^2(b^2-a^2)}+\dfrac{a}{2(a+b)}-\dfrac{a}{2(b-a)}$.

5. Solve the system $\dfrac{2}{a}(x-a)=\dfrac{2}{b}(y-b)=\dfrac{x+y}{a+b}$.

6. Solve the system
$2x+y-z=15,\ x+2y-z=16,\ 3x+4y+2z=80.$

7. Multiply $1-\sqrt{-1}$ by $1+\sqrt{-2}$.

8. Solve the equation $x^{\frac{1}{3}}+2=\dfrac{8}{\sqrt[3]{x}}$.

9. Solve the system $x^3+y^3+z^3=3xyz, x-a=y-b=z-c$.

10. Find two numbers such that the product of their difference and the greater number is 15, and the difference between twice the greater and the less equals the quotient of 16 divided by the less.

F

1. Factor the expression $6(a^2+3)-21a$.

2. Find the h.c.f. of $6y^4+y^3-y$ and $4y^3-6y^2-4y+3$.

3. Find the l.c.m. of $(a^4+a^2+1)^2, (a^3-1)^2, a^3+1, (a^2-1)^4$.

4. Simplify $\dfrac{1}{x+2}-\dfrac{2}{(x+2)(x+4)}+\dfrac{2}{(x+2)(x+4)(x+6)}$.

5. Solve the system $\dfrac{p}{x}+\dfrac{q}{y}=1,\ \dfrac{q}{x}+\dfrac{p}{y}=1$.

6. Solve the system
$$\begin{aligned}5x-6y+4z&=15,\\ 7x+4y-3z&=19,\\ 2x+y+6z&=46.\end{aligned}$$

7. Rationalize the denominator of $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

8. Solve the equation $5+\sqrt[3]{x}=\dfrac{24}{x^{\frac{1}{3}}}$.

9. Solve the system $2x^2-y(x-y)=16, 2y^2+x(x+y)=44$.

10. A sheet of cardboard contains 160 sq. in. If it were 3 in. wider and 4 in. longer, it would contain 260 sq. in. Required its dimensions.

G

1. Factor the expression
$6x^2y+3x^2+12xy^2+12xy+3x+12y^2+6y$.

2. Find the h. c. f. of
$12a^2-15ab+3b^2$ and $6a^3-6a^2b+2ab^2-2b^3$.

3. Find the l. c. m. of a^2-9, a^2-5a+6, $12a-36$.

4. Simplify the expression $\dfrac{x^2+\frac{5}{2}xy+y^2}{x^2+\frac{3}{2}xy-y^2}$.

5. Solve the system $\dfrac{x}{p}+\dfrac{y}{q}=1$, $\dfrac{x}{3p}+\dfrac{y}{6q}=\dfrac{2}{3}$.

6. Solve the system $\dfrac{2}{x}+\dfrac{1}{y}=\dfrac{3}{z}$, $\dfrac{3}{z}-\dfrac{2}{y}=2$, $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{4}{3}$.

7. Extract the square root of $4\sqrt{-1}-3$.

8. Solve the equation $\frac{1}{4}x^{\frac{1}{4}}=1-\dfrac{1}{x^{\frac{1}{4}}}$.

9. Solve the system $x^5+y^5=275$, $x+y=5$.

10. A cistern can be filled by the first of two pipes in 2 hrs. less time than by the other. If both are open together, it is filled in $1\frac{7}{8}$ hrs. How long will it take each pipe alone to fill it?

H

1. Factor the expression

$$ab(b^2 - a^2) + bc(c^2 - b^2) + ca(a^2 - c^2).$$

2. Find the h. c. f. of $2p^5 - 11p^2 - 9$ and $4p^5 + 11p^4 + 81$.

3. Find the l. c. m. of $1 + x + x^2$, $x^2 - x + 1$, $x^2 + x^4 + 1$.

4. Solve the equation $x^3 = 56 + x\sqrt{x}$.

5. Extract the square root of $2\sqrt{6} - 5$.

6. Solve the system $169x^2 + 2y^2 = 177$, $4y^2 - 13x^2 = 3$.

7. Find the value of $\dfrac{x + y - 1}{x - y + 1}$ when

$$x = \frac{m + 1}{mn + 1}, \text{ and } y = \frac{mn + m}{mn + 1}.$$

8. Solve the system $\dfrac{x}{p + q} + \dfrac{y}{p - q} = 2p$, $\dfrac{x - y}{4pq} = 1$.

9. Solve the system $\dfrac{1}{x} + \dfrac{1}{y} = 1$, $\dfrac{1}{z} + \dfrac{1}{x} = 2$, $\dfrac{1}{y} + \dfrac{1}{z} = 1.5$.

10. A man buys a horse for a certain sum and then sells it for \$144. He gains by the transaction as many per cent as the horse cost in dollars. Required the cost of the horse.

Beman, W.W.
Academic algebra.

NAME	DATE	NAME	DATE